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Witten's Invariants of Rational Homology Spheres at Prime Values of K and Trivial Connection Contribution.

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Abstract

We establish a relation between the coefficients of asymptotic expansion of the trivial connection contribution to Witten's invariant of rational homology spheres and the invariants that T. Ohtsuki extracted from Witten's invariant at prime values of K. We also rederive the properties of prime K invariants discovered by H. Murakami and T. Ohtsuki. We do this by using the bounds on Taylor series expansion of the Jones polynomial of algebraically split links, studied in our previous paper. These bounds are enough to prove that Ohtsuki's invariants are of finite type. The relation between Ohtsuki's invariants and trivial connection contribution is verified explicitly for lens spaces and Seifert manifolds.

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1 Introduction

Witten's invariant of 3d manifolds defined in [1] by a path integral over the $SU(2)$ connections A_μ on a 3d manifold M

$$Z(M;k) = \int [\mathcal{D}A_\mu] e^{\frac{ik}{2\pi} S_{\rm CS}[A_\mu]}, \tag{1.1}$$

$$S_{\rm CS} = \frac{1}{2}\,{\rm Tr}\,\epsilon^{\mu\nu\rho} \int_M d^3x \left(A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right) \tag{1.2}$$

($k \in \mathbb{Z}$, Tr is the trace taken in the fundamental representation of $SU(2)$) can also be calculated combinatorially with the help of the surgery formula. Let M be a 3d manifold constructed by $(p_j, 1)$ surgeries on the components $\mathcal{L}_j$ of an N-component link $\mathcal{L}$ in S^3. A $(p,1)$ surgery means that the meridian of the tubular neighborhood is glued to the parallel plus p meridians on the boundary of the knot complement (in other words, a Dehn's surgery is performed on a knot with framing number p). The invariant of M can be expressed in terms of the framing independent colored Jones polynomial $J_{\alpha_1,\ldots,\alpha_N}(\mathcal{L};k)$ of the link $\mathcal{L}$:

$$\begin{aligned} \frac{Z(M;k)}{Z(S^3;k)} = {} & \left(\frac{2}{K}\right)^{\frac{N}{2}} \exp\left[\frac{3}{4}\pi i \left(\frac{2}{K} - 1\right) \sum_{j=1}^{N} {\rm sign}\,(p_j)\right] \\ & \times \sum_{\substack{0\leq\alpha_j\leq K-1 \\ (1\leq j\leq N)}} J_{\alpha_1,\ldots,\alpha_N}(\mathcal{L};k) \exp\left(\frac{i\pi}{2K}\sum_{j=1}^{N} p_j(\alpha_j^2 - 1)\right) \prod_{j=1}^{N} \sin\left(\frac{\pi}{K}\alpha_j\right), \end{aligned} \tag{1.3}$$

$$Z(S^3;k) = \sqrt{\frac{2}{K}} \sin\frac{\pi}{K}, \qquad K = k+2. \tag{1.4}$$

The Jones polynomial $J_{\alpha_1,\ldots,\alpha_N}(\mathcal{L};k)$ is normalized in such a way that it is multiplicative for unlinked links and $J(\text{empty link};k) = 1$, $J_\alpha(\text{unknot};k) = \frac{\sin\left(\frac{\pi}{K}\alpha\right)}{\sin\left(\frac{\pi}{K}\right)}$.

Although N. Reshetikhin and V. Turaev proved [2] that eq. (1.3) indeed defines an invariant of M (*i.e.* the *l.h.s.* of eq. (1.3) is invariant under Kirby moves), the topological origin of this invariant remains somewhat obscure. The question is: which of the "classical" topological invariants of M are contained inside $Z(M;k)$? Two distinct approaches to this problem have been tried. The first one is to study $Z(M;k)$ for some particular values of K. R. Kirby and P. Melvin discovered [3] that if K is odd, then $Z(M;k)$ is proportional to

$Z(M;1)$:

$$\frac{Z(M;k)}{Z(S^3;k)} = Z'(M;k) \times \begin{cases} \frac{Z(M;1)}{Z(S^3;1)}, & \text{if } K = -1 \pmod 4 \\ \frac{Z(M;1)}{Z(S^3;1)}, & \text{if } K = 1 \pmod 4, \end{cases} \tag{1.5}$$

If M is a rational homology sphere ($\mathbb{Z}$HS), then $Z(M;1) = Z(S^3;1)$ so that $Z'(M;k) = \frac{Z(M;k)}{Z(S^3;k)}$.

The new invariant $Z'(M;k)$ can be calculated by the following surgery formula

$$\begin{aligned} Z'(M;k) = K^{-\frac{N}{2}} \exp\left[-\frac{\pi i}{4}\left(3 + \kappa - \frac{6}{K}\right)\sum_{j=1}^{N} \operatorname{sign}(p_j)\right] \\ \times \sideset{}{'}\sum_{\substack{-K \le \alpha_j \le K \\ \alpha_j \in 2\mathbb{Z}+1 \\ (1 \le j \le N)}} J_{\alpha_1,\ldots,\alpha_N}(\mathcal{L};k) \exp\left(\frac{i\pi}{2K}\sum_{j=1}^{N} p_j(\alpha_j^2 - 1)\right) \prod_{j=1}^{N} \sin\left(\frac{\pi}{K}\alpha_j\right), \end{aligned} \tag{1.6}$$

here

$$\kappa = \begin{cases} 1 & \text{if } K = -1 \pmod 4 \\ -1 & \text{if } K = 1 \pmod 4, \end{cases} \tag{1.7}$$

while $\sum'$ means that we add an extra factor of $\frac{1}{2}$ to the terms corresponding to the boundary values of summation index ($\alpha_j = \pm K$ in this case). We changed slightly the original formula of [3]: instead of taking a sum over $1 \le \alpha_j \le \frac{K-1}{2}$ we sum over odd α_j between 1 and $K-1$. This allows us to get rid of some phase factors. We also double the range of summation to $1 - K \le \alpha_j \le K - 1$ by using the fact that $J_{\alpha_1,\ldots,\alpha_N}(\mathcal{L};k)$ is an odd function of its indices (we use the $2K$ periodicity in α_j in order to extend $J_{\alpha_1,\ldots,\alpha_N}(\mathcal{L};k)$ to negative values of α_j). Note that the whole summand of eq. (1.6) has a periodicity of $2K$.

S. Garoufalidis [4] used nice properties of the gaussian sum $\sum_{\alpha=0}^{K-1} \exp\left(\frac{2\pi i}{K}\alpha^2\right)$ for prime values of K in order to study Witten's invariant of lens spaces and Seifert manifolds. H. Murakami and T. Ohtsuki [5]–[8] carried out a detailed study of the invariant $Z'(M;k)$ of rational homology spheres (RHS) for prime K.

Theorem 1.1 (H. Murakami, [5], [6]) *For a RHS M and a prime $K > 2$*

$$Z'(M;k) \in \mathbb{Z}[\check{q}], \qquad \check{q} = e^{\frac{2\pi i}{K}}, \tag{1.8}$$

$\mathbb{Z}[\check{q}]$ *being a cyclotomic ring.*

We need more notations in order to present the results of Ohtsuki's papers [7], [8]. We introduce a new variable

$$x = q - 1. \tag{1.9}$$

A polynomial from $\mathbb{Z}[\check{q}]$ can be reexpressed as a polynomial in x with integer coefficients. It is defined modulo the polynomial

$$\frac{(1+x)^K - 1}{x} = \sum_{n=0}^{K-1} \frac{K(K-1)\cdots(K-n)}{(n+1)!} x^n, \tag{1.10}$$

which is identically equal to zero for $x = e^{\frac{2\pi i}{K}} - 1$. All the coefficients of this polynomial except the one at x^{K-1}, are divisible by K. As a result, all the coefficients at x^n, $n \leq K-2$ for a polynomial of x coming from $\mathbb{Z}[\check{q}]$ are well defined modulo K. We will limit our attention to the powers of x up to $x^{\frac{K-1}{2}}$. They are all well defined as elements of $\mathbb{Z}_K$ if $K \geq 3$. Thus there is a homomorphism of rings:

$$^\diamond : \quad \mathbb{Z}[\check{q}] \to \mathbb{Z}'_K[x] \stackrel{\text{def.}}{=} \mathbb{Z}_K[x]/x^{\frac{K+1}{2}} \mathbb{Z}_K[x]. \tag{1.11}$$

There is another homomorphism from polynomials (of maybe infinite degree) of x with rational coefficients to $\mathbb{Z}'_K[x]$:

$$^\vee : \quad \mathbb{Q}[[x]] \to \mathbb{Z}'_K[x]. \tag{1.12}$$

The action of the operation $^\vee$ on rational numbers was introduced in relation to Witten's invariants at prime values of K by S. Garoufalidis [4]:

$$^\vee : \quad \mathbb{Q} \to \mathbb{Z}_K, \qquad \left(\frac{p}{q}\right)^\vee = pq^*, \tag{1.13}$$

here q^* is the inverse of q modulo K: $qq^* = 1 \pmod{K}$. The homomorphism $^\vee$ acts on polynomials (infinite series) by removing all powers of x higher than $\frac{K-1}{2}$ and converting the remaining coefficients to $\mathbb{Z}_K$.

Now we can present (a slightly stronger version of) Ohtsuki's results:

Theorem 1.2 (T. Ohtsuki [7], [8]) *For any RHS M there exists a sequence of rational numbers $\lambda_n(M) \in \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{2n}, \frac{1}{|H_1(M,\mathbb{Z})|}\right] \subset \mathbb{Q}$, $n \geq 0$ so that for any prime number K such that $|H_1(M,\mathbb{Z})| \neq 0 \pmod K$*

$$\left[|H_1(M,\mathbb{Z})| \left(\frac{|H_1(M,\mathbb{Z})|}{K}\right) Z'(M;k)\right]^{\diamond} = \left[\sum_{n=0}^{\infty} \lambda_n(M) x^n\right]^{\vee}, \tag{1.14}$$

here $\left(\frac{\cdot}{K}\right)$ is the Legendre symbol.

We have slightly modified the theorem of [8]: Ohtsuki required that $K > |H_1(M,\mathbb{Z})|$, he estimated that $\lambda_n(M) \in \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{2n+1}, \frac{1}{|H_1(M,\mathbb{Z})|}\right] \subset \mathbb{Q}$, $n > 0$ and he used $\mathbb{Z}'_K[x] = \mathbb{Z}_K[x]/x^{\frac{K-1}{2}}\mathbb{Z}[x]$ instead of (1.11) (in other words, he did not fix the coefficient at $x^{\frac{K-1}{2}}$). Murakami showed that

$$\lambda_0 = 1, \qquad \lambda_1 = \lambda_{\mathrm{CW}}, \tag{1.15}$$

here λ_{CW} is the Casson-Walker invariant of RHS .

The second approach to the search of the topological meaning of Witten's invariant $Z(M;k)$ is based on the path integral representation (1.1). According to quantum field theory, this integral can be calculated by stationary phase approximation when $K \to \infty$. The invariant is presented as a sum of contributions coming from connected components c of the moduli space of flat connections on the manifold M:

$$Z(M;k) = \sum_c Z^{(c)}(M;k). \tag{1.16}$$

Each contribution has a general form

$$Z^{(c)}(M;k) = \left(\frac{4\pi^2}{K}\right)^{\frac{N_{\mathrm{zero}}}{2}} \frac{1}{\mathrm{Vol}(H_c)} \exp\left[2\pi i k S^{(c)}_{\mathrm{CS}} + \sum_{n=0}^{\infty} S^{(c)}_n(M) \left(\frac{i\pi}{K}\right)^n\right], \tag{1.17}$$

here H_c is the isotropy group, $N_{\mathrm{zero}} = \dim H^0_c - \dim H^1_c$, $H^{0,1}_c$ being the cohomologies of 0,1-forms taking values in the adjoint $su(2)$ bundle, $S^{(c)}_{\mathrm{CS}}$ is the Chern-Simons action and $\sum_{n=0}^{\infty} S^{(c)}_n(M) \left(\frac{i\pi}{K}\right)^n$ is an asymptotic series. The coefficients $S^{(c)}_n(M)$ are called $(n+1)$-loop corrections. They might be related to "classical" topological invariants of M. Indeed, the 1-loop correction $S^{(c)}_0$ is related to the Reidemeister-Ray-Singer torsion. An attempt to relate

the asymptotic properties of the surgery formula (1.3) for lens spaces and Seifert manifolds to the quantum field theory predictions of eqs. (1.16), (1.17) was initiated by D. Freed and R. Gompf [9] and carried out further by L. Jeffrey [10], S. Garoufalidis [4] and also in the papers [11, 12]. A complete agreement between the surgery formula and 1-loop predictions was observed.

If the manifold M is a RHS , then the trivial connection is a separate point in the moduli space of flat connections. According to quantum field theory, its contribution is of the form

$$Z^{(\mathrm{tr})}(M;k) = \frac{\sqrt{2}\pi}{K^{\frac{3}{2}}|H_1(M,\mathbb{Z})|^{\frac{3}{2}}} \exp\left[\sum_{n=1}^{\infty} S_n(M)\left(\frac{i\pi}{K}\right)^n\right]. \tag{1.18}$$

A representation of the coefficients $S_n(M)$ in terms of $(n+1)$-loop Feynman diagrams was carried out by S. Axelrod and I. Singer [13], M. Kontsevich [14], C. Taubes [15] and others. We studied how the trivial connection contribution can be extracted from the surgery formula (1.3). We derived a knot surgery formula [16] and a link surgery formula [17] for it. The knot formula allowed us to show that

$$S_1(M) = 6\lambda_{\mathrm{CW}}. \tag{1.19}$$

The link surgery formula of [17] was much less explicit than the knot formula of [16], because it did not express $Z^{(\mathrm{tr})}(M;k)$ directly in terms of derivatives of the Jones polynomial $J_{\alpha_1,\ldots,\alpha_N}(\mathcal{L};k)$. However we derived an explicit surgery formula [18] for algebraically split links (ASL).

In this paper we are going to prove the following:

Proposition 1.1 *Ohtsuki's invariants $\lambda_n(M)$ of eq. (1.14) and loop corrections to the trivial connection contribution $S_n(M)$ of eq. (1.18) can be expressed in terms of each other through the following relation*

$$\sum_{n=0}^{\infty} \lambda_n(M)x^n = \exp\left[\sum_{n=1}^{\infty}(S_n(M)-S_n(S^3))\left(\frac{i\pi}{K}\right)^n\right] \equiv \frac{\left(\frac{\pi}{K}\right)}{\sin\left(\frac{\pi}{K}\right)} \exp\left[\sum_{n=1}^{\infty} S_n(M)\left(\frac{i\pi}{K}\right)^n\right] \tag{1.20}$$

by substituting either

$$x = e^{\frac{2\pi i}{K}} - 1 = \frac{2\pi i}{K}\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left(\frac{2\pi i}{K}\right)^n, \tag{1.21}$$

or

$$\frac{i\pi}{K}=\frac{1}{2}\log(1+x)=\frac{x}{2}\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n+1}. \tag{1.22}$$

In other words, we will show that

$$\left[|H_1(M,\mathbb{Z})|\left(\frac{|H_1(M,\mathbb{Z})|}{K}\right)Z'(M;k)\right]^{\diamond}=\left[|H_1(M,\mathbb{Z})|^{\frac{3}{2}}\frac{Z^{(\mathrm{tr})}(M;k)}{Z(S^3;k)}\right]^{\vee}. \tag{1.23}$$

In process of doing this we will rederive the results of [5]-[8] maybe in a somewhat more explicit way.

Our proof of the Proposition 1.1 will be based on the following two propositions derived in [19] and [18] by using some "physical" considerations:

Proposition 1.2 *Let $\mathcal{L}$ be an algebraically split link (ASL) in S^3. Then its framing-independent colored Jones polynomial has the following Taylor series expansion in powers of K:*

$$J_{\alpha_1,\ldots,\alpha_N}(\mathcal{L};k)=\left(\prod_{j=1}^{N}\alpha_j\right)\sum_{n=0}^{\infty}\sum_{m\le\frac{3}{4}n}D_{m,n}(\alpha_1,\ldots,\alpha_N)\left(\frac{i\pi}{K}\right)^n, \tag{1.24}$$

here $D_{m,n}(\alpha_1,\ldots,\alpha_N)$ are even homogeneous polynomials of degree $2m$:

$$D_{m,n}(\alpha_1,\ldots,\alpha_N)=\sum_{\substack{m_1,\ldots,m_N\ge0\\ m_1+\cdots+m_N=m}}D^{(m,n)}_{m_1,\ldots,m_N}\alpha_1^{2m_1}\cdots\alpha_N^{2m_N} \tag{1.25}$$

and

$$m_j\le n-m,\qquad 1\le j\le N. \tag{1.26}$$

Proposition 1.3 *Let M be a rational homology sphere (RHS) constructed by $(p_j,1)$ surgeries on the components of an N-component link $\mathcal{L}$ in S^3. Then the loop corrections to the trivial connection contribution (1.18) to Witten's invariant of M are given by the formula:*

$$\begin{aligned}&\exp\left[\sum_{n=1}^{\infty}(S_n(M)-S_n(S^3))\left(\frac{i\pi}{K}\right)^n\right] \qquad (1.27)\\ &=\exp\left[\frac{3}{4}\pi i\left(\frac{2}{K}-1\right)\sum_{j=1}^{N}\mathrm{sign}\,(p_j)\right]\exp\left[-\frac{i\pi}{2K}\sum_{j=1}^{N}\left(p_j+\frac{1}{p_j}\right)\right]\frac{i^N}{(2K)^{\frac{N}{2}}}\left|\prod_{j=1}^{N}p_j\right|^{\frac{3}{2}}\\ &\times\int_{\substack{-\infty\\ [\alpha_j=0]}}^{+\infty}d\alpha_1\cdots d\alpha_N\,\exp\left(\frac{i\pi}{2K}\sum_{j=1}^{N}p_j\alpha_j^2\right)J_{\alpha_1+\frac{1}{p_1},\ldots,\alpha_N+\frac{1}{p_N}}(\mathcal{L};k).\end{aligned}$$

The symbol $\int\limits_{-\infty \atop [\alpha_j=0]}^{+\infty}$ *means that the integral has to be calculated in the following way: first, an expansion (1.24) has to be substituted and then the gaussian integrals over* α_j *have to be calculated for each polynomial* $\left(\prod_{j=1}^N \left(\alpha_j + \frac{1}{p_j}\right)\right) D_{m,n}\left(\alpha_1 + \frac{1}{p_1}, \ldots, \alpha_N + \frac{1}{p_N}\right)$ *separately (for more details see [18]).*

The Proposition 1.2 is essential for all our calculations. The Proposition 1.4 is needed only for the derivation of eq. (1.23). In other words, we could use eq. (1.27) as a definition of $S_n(M)$ and then prove eq. (1.20) which amounts to proving the Theorem 1.2.

In Section 2 we modify the surgery formula (1.6) and prove the Theorem 1.1. Our main tool is the observation that the gaussian sum $\sum_{\alpha=0}^{K-1} \check{q}^{\alpha^2}$ is proportional to $x^{\frac{K-1}{2}}$, while the sum $\sum_{\alpha=0}^{K-1} \check{q}^{\alpha^2}\alpha^{2m}$ for $m \leq \frac{K-1}{2}$ is only proportional to $x^{\frac{K-1}{2}-m}$. This is similar to the behavior of gaussian integrals: each two extra powers of α in the integral $\int_{-\infty}^{+\infty} e^{\frac{2\pi i}{K}\alpha^2}\alpha^{2m}\, d\alpha$ bring a power of K to denominator. In Section 3 we prove the Proposition 1.1 and thus also the Theorem 1.2. We use again the similarities of the formulas for $\sum_{\alpha=0}^{K-1} \check{q}^{\alpha^2}\alpha^{2m}$ and $\int_{-\infty}^{+\infty} e^{\frac{2\pi i}{K}\alpha^2}\alpha^{2m}\, d\alpha$. These similarities are due to the fact that both the sum $\sum_{\alpha=0}^{K-1} \check{q}^{\alpha^2+2n\alpha}$ and the integral $\int_{-\infty}^{+\infty} e^{\frac{2\pi i}{K}(\alpha^2+2n\alpha)}\, d\alpha$ can be calculated by completing the square in the exponents. In Section 4 we derive a rational surgery formula for $Z'(M;k)$ which is similar to the formula (4.1) of [10] for the original Witten's invariant $Z(M;k)$. We use this formula to verify the Proposition 1.1 for lens spaces and Seifert manifolds which are rational homology spheres. In Section 5 we discuss the properties of Ohtsuki's invariants $\lambda_n(M)$ as related to the properties of invariants $S_n(M)$ studied in [18].

2 Gaussian Sums and Divisibility in Cyclotomic Ring

We start by modifying the surgery formula (1.6). Since $\frac{1}{4}(\alpha_j^2 - 1) \in \mathbb{Z}$, then

$$\exp\left[\frac{i\pi}{2K}\sum_{j=1}^N p_j(\alpha_j^2 - 1)\right] = \check{q}^{\frac{1}{4}\sum_{j=1}^N p_j(\alpha_j^2-1)} = \check{q}^{4^*\sum_{j=1}^N p_j(\alpha_j^2-1)}. \tag{2.1}$$

Also since $\frac{1}{2}(\alpha_j \pm \mathrm{sign}\,(p_j)) \in \mathbb{Z}$,

$$\sin\left(\frac{\pi}{K}\alpha_j\right) = \frac{i}{2}\sum_{\mu_j=\pm 1}\mu_j e^{-\frac{i\pi}{K}\mu_j\alpha_j} = \frac{i}{2}\sum_{\mu_j=\pm 1}\mu_j \check{q}^{-2^*\mu_j\alpha_j}\,\check{q}^{\left(2^*-\frac{1}{2}\right)\mathrm{sign}(p_j)}. \tag{2.2}$$

The Jones polynomial $J_{\alpha_1,\ldots,\alpha_N}(\mathcal{L};k)$ is odd and $e^{\frac{i\pi}{2K}p_j\alpha_j^2}$ is even as a function of α_j. Therefore we can drop the factor $\frac{1}{2}$ and put $\mu_j = 1$ in eq. (2.2) upon substituting it into eq. (1.6):

$$\begin{aligned} Z'(M;k) \;=\; & K^{-\frac{N}{2}} i^N e^{-\frac{i\pi}{4}(3+\kappa)\sum_{j=1}^N \mathrm{sign}(p_j)}\check{q}^{\left(\frac{1}{4}+2^*\right)\sum_{j=1}^N \mathrm{sign}(p_j)}\check{q}^{-4^*\sum_{j=1}^N p_j} \\ & \times \sideset{}{'}\sum_{\substack{-K\le\alpha_j\le K \\ \alpha_j\in 2\mathbb{Z}+1 \\ (1\le j\le N)}} \check{q}^{\sum_{j=1}^N (4^* p_j\alpha_j^2 - 2^*\alpha_j)} J_{\alpha_1,\ldots,\alpha_N}(\mathcal{L};k). \end{aligned} \tag{2.3}$$

After completing the square

$$4^* p_j\alpha_j^2 - 2^*\alpha_j = 4^* p_j(\alpha_j - p_j^*)^2 - 4^* p_j^* \pmod K, \tag{2.4}$$

we shift the summation variable α_j by p_j^* (we assume that p_j^* is even in order to preserve the parity of α_j, we can always make such choice of p_j^* since K is odd). Then

$$\begin{aligned} Z'(M;k) \;=\; & K^{-\frac{N}{2}} i^N e^{-\frac{i\pi}{4}(3+\kappa)\sum_{j=1}^N \mathrm{sign}(p_j)}\check{q}^{\left(\frac{1}{4}+2^*\right)\sum_{j=1}^N \mathrm{sign}(p_j)}\check{q}^{-4^*\sum_{j=1}^N (p_j+p_j^*)} \\ & \times \sideset{}{'}\sum_{\substack{-K\le\alpha_j\le K \\ \alpha_j\in 2\mathbb{Z}+1 \\ (1\le j\le N)}} \check{q}^{4^*\sum_{j=1}^N p_j\alpha_j^2} J_{\alpha_1+p_1^*,\ldots,\alpha_N+p_N^*}(\mathcal{L};k). \end{aligned} \tag{2.5}$$

Next we use the identities

$$ie^{-\frac{i\pi}{2}\mathrm{sign}(p_j)} = \mathrm{sign}\,(p_j)\,, \tag{2.6}$$

$$\frac{1-\kappa K}{4} = 4^* \tag{2.7}$$

in order to rearrange the phase factors preceding the sum in eq. (2.5):

$$\begin{aligned} Z'(M;k) \;=\; & K^{-\frac{N}{2}} e^{\frac{i\pi}{4}(\kappa-1)\sum_{j=1}^N \mathrm{sign}(p_j)}\left(\prod_{j=1}^N \mathrm{sign}\,(p_j)\right)\check{q}^{4^*\sum_{j=1}^N (3\,\mathrm{sign}(p_j)-p_j-p_j^*)} \\ & \times \sideset{}{'}\sum_{\substack{-K\le\alpha_j\le K \\ \alpha_j\in 2\mathbb{Z}+1 \\ (1\le j\le N)}} \check{q}^{4^*\sum_{j=1}^N p_j\alpha_j^2} J_{\alpha_1+p_1^*,\ldots,\alpha_N+p_N^*}(\mathcal{L};k). \end{aligned} \tag{2.8}$$

Since the values of the Jones polynomial $J_{\alpha_1,\ldots,\alpha_N}(\mathcal{L};k)$ belong to the cyclotomic ring $\mathbb{Z}[\check{q}]$ when all the indices α_j are odd, we can apply to it a combination of Proposition 1.2 and Lemma 2.3 of [5]:

Proposition 2.1 *For odd values of its indices α_j, the unframed colored Jones polynomial of an N-component ASL $\mathcal{L}$ in S^3 can be presented as the following sum:*

$$J_{\alpha_1,\ldots,\alpha_N}(\mathcal{L};k) = \left(\prod_{j=1}^{N}\alpha_j\right)\sum_{n=0}^{(N+1)\frac{K-1}{2}}\sum_{m\leq\frac{3}{4}n}\tilde{D}_{m,n}(\alpha_1,\ldots,\alpha_N)x^n \tag{2.9}$$
$$+x^{(N+1)\frac{K-1}{2}+1}J^{(\mathrm{res})}_{\alpha_1,\ldots,\alpha_N}(\mathcal{L};k), \qquad J^{(\mathrm{res})}_{\alpha_1,\ldots,\alpha_N}(\mathcal{L};k)\in\mathbb{Z}[\check{q}],$$

here $\tilde{D}_{m,n}(\alpha_1,\ldots,\alpha_N)$ are homogeneous polynomials of degree $2m$:

$$\tilde{D}_{m,n}(\alpha_1,\ldots,\alpha_N) = \sum_{\substack{m_1,\ldots,m_N\geq 0\\ m_1+\cdots+m_N=m}}\tilde{D}^{(m,n)}_{m_1,\ldots,m_N}\alpha_1^{2m_1}\cdots\alpha_N^{2m_N}, \tag{2.10}$$
$$m_j\leq n-m, \tag{2.11}$$

and the polynomials $\left(\prod_{j=1}^{N}\alpha_j\right)\sum_{m\leq\frac{3}{4}n}\tilde{D}_{m,n}(\alpha_1,\ldots,\alpha_N)$ take integer values when α_j are odd.

The latter property of the polynomials $\tilde{D}_{m,n}$ allows us to express them in terms of "binomial coefficient" polynomials:

$$\left(\prod_{j=1}^{N}\alpha_j\right)\sum_{m\leq\frac{3}{4}n}\tilde{D}_{m,n}(\alpha_1,\ldots,\alpha_N) = \sum_{\substack{1\leq m_j\leq 2\bar{m}_j+1\\ (1\leq j\leq N)}}C^{(n)}_{m_1,\ldots,m_N}P_{m_1}\left(\frac{\alpha_1-1}{2}\right)\cdots P_{m_N}\left(\frac{\alpha_N-1}{2}\right),$$
$$C^{(n)}_{m_1,\ldots,m_N}\in\mathbb{Z}, \tag{2.12}$$

here

$$P_m(\alpha) = \frac{\alpha(\alpha-1)\cdots(\alpha-m+1)}{m!} \tag{2.13}$$

and $\bar{m}_j$ are the maximum values of m_j in the representation (2.10) of all the polynomials $\tilde{D}_{m,n}$ appearing in the *l.h.s.* of eq. (2.12).

The following proposition is a reflection of the inequality (2.11) for the representation (2.12):

Proposition 2.2 *There is an upper bound on the indices of the coefficients* $C^{(n)}_{m_1,\dots,m_N}$ *of eq. (2.12):*

$$\left[\frac{m_j}{2}\right] \le n - \sum_{i=1}^{N} \left[\frac{m_i}{2}\right], \tag{2.14}$$

here $\left[\frac{m}{2}\right]$ *denotes the integer part of* $\frac{m}{2}$.

The proof is completely similar to that of the Proposition 3.4 of [18]. Suppose that there is a coefficient $C^{(n)}_{m_1,\dots,m_N}$ for which (2.14) is not true, say, for m_1. If all indices m_j of $C^{(n)}_{m_1,\dots,m_N}$ are odd, then the highest degree monomial of the corresponding polynomial

$$C^{(n)}_{m_1,\dots,m_N} P_{m_1}\left(\frac{\alpha_1 - 1}{2}\right) \cdots P_{m_N}\left(\frac{\alpha_N - 1}{2}\right) \tag{2.15}$$

violates the inequality (2.11). Therefore it has to be canceled by monomials of other polynomials

$$C^{(n)}_{m'_1,\dots,m'_N} P_{m'_1}\left(\frac{\alpha_1 - 1}{2}\right) \cdots P_{m'_N}\left(\frac{\alpha_N - 1}{2}\right) \tag{2.16}$$

for which

$$m'_j \ge m_j, \quad 1 \le j \le N, \qquad \sum_{j=1}^{N} m'_j > \sum_{j=1}^{N} m_j. \tag{2.17}$$

If some m_j are even, then the highest degree monomial of the polynomial (2.15) is incompatible with the structure of the *l.h.s.* of eq. (2.12) and it also has to be canceled. The inequalities (2.17) show that the index m_1 of the polynomials (2.15) again violates (2.14), so we need to go to higher values of $\sum_{j=1}^{N} m_j$ for further cancelation. Since $\sum_{j=1}^{N} m_j \le n$, this process can not be completed. This contradiction proves the proposition.

Now we begin to prove the Theorem 1.1. Following [5], we use the relations

$$\sum_{\alpha=0}^{K-1} \check{q}^{\alpha^2} = e^{i\frac{\pi}{4}(1-\kappa)}\sqrt{K}, \tag{2.18}$$

$$\sum_{\alpha=0}^{K-1} \check{q}^{\alpha^2} = x^{\frac{K-1}{2}} u^{-1}, \qquad u,\, u^{-1} \in \mathbb{Z}[\check{q}] \tag{2.19}$$

in order to present $K^{-\frac{N}{2}}$ in the following form:

$$K^{-\frac{N}{2}} = e^{i\frac{\pi}{4}N(1-\kappa)} \frac{u^N}{x^{N\frac{K-1}{2}}}. \tag{2.20}$$

Substituting this expression into eq. (2.8) we find that

$$Z'(M;k) = e^{\frac{i\pi}{4}(\kappa-1)\sum_{j=1}^{N}(\mathrm{sign}(p_j)-1)}\left(\prod_{j=1}^{N}\mathrm{sign}\,(p_j)\right)\check{q}^{4^*\sum_{j=1}^{N}(3\,\mathrm{sign}(p_j)-p_j-p_j^*)}$$
$$\times\frac{u^N}{x^{N\frac{K-1}{2}}}\sideset{}{'}\sum_{\substack{-K\le\alpha_j\le K\\ \alpha_j\in 2\mathbb{Z}+1\\ (1\le j\le N)}}\check{q}^{4^*\sum_{j=1}^{N}p_j\alpha_j^2}J_{\alpha_1+p_1^*,\ldots,\alpha_N+p_N^*}(\mathcal{L};k). \tag{2.21}$$

Since $\frac{1}{4}(\kappa-1)(\mathrm{sign}\,(p_j)-1)\in\mathbb{Z}$, we conclude that to prove Theorem 1.1 it is enough to show that

$$\sideset{}{'}\sum_{\substack{-K\le\alpha_j\le K\\ \alpha_j\in 2\mathbb{Z}+1\\ (1\le j\le N)}}\check{q}^{4^*\sum_{j=1}^{N}p_j\alpha_j^2}J_{\alpha_1+p_1^*,\ldots,\alpha_N+p_N^*}(\mathcal{L};k) = x^{N\frac{K-1}{2}}w, \qquad w\in\mathbb{Z}[\check{q}]. \tag{2.22}$$

We will substitute the expansion (2.9) and check the property (2.22) for every polynomial $\left(\prod_{j=1}^{N}\alpha_j\right)\sum_{m\le\frac{3}{4}n}\tilde{D}_{m,n}(\alpha_1,\ldots,\alpha_N)$ separately. The remainder term $x^{(N+1)\frac{K-1}{2}}J^{(\mathrm{res})}_{\alpha_1,\ldots,\alpha_N}(\mathcal{L};k)$ obviously satisfies eq. (2.22). Moreover, for some $w\in\mathbb{Z}[\check{q}]$,

$$\left[\frac{1}{x^{N\frac{K-1}{2}}}\sideset{}{'}\sum_{\substack{-K\le\alpha_j\le K\\ \alpha_j\in 2\mathbb{Z}+1\\ (1\le j\le N)}}\check{q}^{4^*\sum_{j=1}^{N}p_j\alpha_j^2}J^{(\mathrm{res})}_{\alpha_1,\ldots,\alpha_N}(\mathcal{L};k)\,x^{(N+1)\frac{K-1}{2}+1}\right]^{\diamond} = \left[x^{\frac{K+1}{2}}w\right]^{\diamond}=0, \tag{2.23}$$

so that we can neglect the contribution of this term in all further calculations.

To estimate the contribution of a polynomial $\left(\prod_{j=1}^{N}\alpha_j\right)\sum_{m\le\frac{3}{4}n}\tilde{D}_{m,n}(\alpha_1,\ldots,\alpha_N)\,x^n$ we need the following simple lemma:

Lemma 2.1 *For $p,m\in\mathbb{Z}$, $m\ge 0$*

$$\sideset{}{'}\sum_{\substack{-K\le\alpha\le K\\ \alpha\in 2\mathbb{Z}+1}}\check{q}^{p\alpha^2}\alpha^{2m} = x^{\max\{0,\frac{K-1}{2}-m\}}w, \qquad w\in\mathbb{Z}[\check{q}]. \tag{2.24}$$

The lemma needs a proof only for $m<\frac{K-1}{2}$. To prove that an element $u\in\mathbb{Z}[\check{q}]$ is divisible by x^n, $n\le K-1$ one may present it as an integer coefficient polynomial of x and check that the coefficients in front of all $x^{n'}$, $n'<n$ are divisible by K. We substitute $\check{q}=x+1$ in eq. (2.24) and express the powers of $\check{q}$ in terms of "binomial" polynomials

$$\sideset{}{'}\sum_{\substack{-K\le\alpha\le K\\ \alpha\in 2\mathbb{Z}+1}}\check{q}^{p\alpha^2}\alpha^{2m} = \sum_{n\ge 0}\sideset{}{'}\sum_{\substack{-K\le\alpha\le K\\ \alpha\in 2\mathbb{Z}+1}}P_n(p\alpha^2)\alpha^{2m}x^n = \sum_{n\ge 0}\sum_{0\le m'\le n}\sideset{}{'}\sum_{\substack{-K\le\alpha\le K\\ \alpha\in 2\mathbb{Z}+1}}\frac{C_{n,m'}}{n!}p^{2m'}\alpha^{2(m+m')}x^n, \tag{2.25}$$

here

$$\alpha(\alpha-1)\cdots(\alpha-n+1)=\sum_{m'\le n}C_{n,m'}\alpha^{m'},\qquad C_{n,m'}\in\mathbb{Z}. \tag{2.26}$$

It is well known in number theory that

$$\sum_{\substack{-K\le\alpha\le K\\ \alpha\in2\mathbb{Z}+1}}{}^{'}\ \alpha^{2m}=0\ (\mathrm{mod}\,K)\qquad\text{for}\quad 0\le m<K-2. \tag{2.27}$$

Therefore the numerator of the contribution of a term $\frac{C_{n,m'}}{n!}p^{2m'}\alpha^{2(m+m')}x^n$ will be divisible by K for all $m+m'<\frac{K-1}{2}\le K-2$, that is, for all $n<\frac{K-1}{2}-m$. The denominator $n!$ is harmless, because since $n<K$, it is not divisible by K and can not cancel the factors of K coming from the sum over α. This proves the lemma.

This lemma can be easily generalized to the "binomial" polynomials (2.13):

Lemma 2.2 *For $p,p',m\in\mathbb{Z}$, $p'\in2\mathbb{Z}$, $m\ge0$*

$$\sum_{\substack{-K\le\alpha\le K\\ \alpha\in2\mathbb{Z}+1}}{}^{'}\ \check{q}^{p\alpha^2}P_m\left(\frac{\alpha+p'-1}{2}\right)=x^{\max\{0,\frac{K-1}{2}-[\frac{m}{2}]\}}\,w,\qquad w\in\mathbb{Z}[\check{q}]. \tag{2.28}$$

The proof is similar to that of the previous lemma. The choice of summation range for α obviates the fact that odd powers of α in $P_m\left(\frac{\alpha+p'-1}{2}\right)$ can be ignored. The numerators of the contributions of even powers of α are divisible by K. The coefficients of P_m have a denominator $m!$, but we may assume that $m<K-1$ (otherwise eq. (2.28) is obvious) so that the denominator does not cancel the factors of K.

Let $\bar{m}(n)$ be the maximum value of m appearing in the *l.h.s.* of eq. (2.12). Then for every coefficient $C^{(n)}_{m_1,\ldots,m_N}$ from the *r.h.s.* of that equation

$$m_1+\cdots+m_N\le\bar{m}(n). \tag{2.29}$$

Therefore we can combine eqs. (2.12) and (2.28) into the following estimate of the contribution of polynomials $\tilde{D}_{m,n}$ to the sum (2.22):

$$\sum_{\substack{-K\le\alpha\le K\\ \alpha\in2\mathbb{Z}+1}}{}^{'}\ \check{q}^{4^*\sum_{j=1}^N p_j\alpha_j^2}\left(\prod_{j=1}^N\left(\alpha_j+p_j^*\right)\right)\sum_{m\le\frac{3}{4}n}\tilde{D}_{m,n}(\alpha_1+p_1^*,\ldots,\alpha_N+p_N^*)\,x^n$$
$$=x^{N\frac{K-1}{2}+n-\bar{m}(n)}\,w,\qquad w\in\mathbb{Z}[\check{q}]. \tag{2.30}$$

Since $\bar{m}(n) \leq \frac{3}{4}n \leq n$, this estimate is enough to prove eq. (2.22) and also the Theorem 1.1. Note that the proof required only a weaker bound $m \leq n$ for $\tilde{D}_{m,n}$ rather than a stronger bound $m \leq \frac{3}{4}n$ of [18]. However the bound $m \leq \frac{3}{4}n$ is necessary to prove that only a finite number of polynomials $\tilde{D}_{m,n}$ contribute to the coefficients of $x^{n'}$, $n' \leq \frac{K-1}{2}$ in the expansion of $[Z'(M;k)]^{\diamond}$. Indeed, since $\bar{m}(n) \leq \frac{3}{4}n$, then $n - \bar{m}(n) \geq \frac{1}{4}n$ and eq. (2.30) suggests that we may limit our attention to only those polynomials (2.15) for which

$$n \leq 2(K-1). \tag{2.31}$$

3 Gaussian Sums and Integrals

We are going to derive a surgery formula for $[Z'(M;k)]^{\diamond}$ which would express it in terms of the derivatives $\tilde{D}_{m,n}$ of the colored Jones polynomial. As we will see, this requires a calculation of the gaussian sum

$$G(p,q;m) = \frac{e^{i\frac{\pi}{4}(\kappa-1)}}{\sqrt{K}} \sideset{}{'}\sum_{\substack{-K \leq \alpha \leq K \\ \alpha \in 2\mathbb{Z}+1}} \check{q}^{q^* p \alpha^2} \alpha^{2m} x^m, \qquad m \leq \frac{K-1}{2}. \tag{3.1}$$

More precisely, we need to find only $[G(p,q;m)]^{\diamond}$. We already know that $G(p,q;m) \in \mathbb{Z}[\check{q}]$.

Proposition 3.1 *The sum of eq. (3.1) is related to the gaussian integral. For $p,q \in \mathbb{Z}$, $p,q \neq 0 \pmod{K}$, $0 \leq m \leq \frac{K-1}{2}$,*

$$\begin{aligned}[G(p,q;m)]^{\diamond} \;=\; \left(\frac{pq^*}{K}\right) \left[e^{-i\frac{\pi}{4}\operatorname{sign}\left(\frac{p}{q}\right)} \left(\frac{2}{K}\right)^{\frac{1}{2}} \left|\frac{p}{q}\right|^{\frac{1}{2}} \int_{-\infty}^{+\infty} d\alpha\, e^{\frac{2\pi i}{K}\frac{p}{q}\alpha^2} \alpha^{2m}\, x^m \right]^{\vee} \\ + x^{\frac{K+1}{2}-m} w, \quad w \in \mathbb{Z}[\check{q}]. \end{aligned} \tag{3.2}$$

To prove the proposition we calculate the following sum:

$$\frac{e^{i\frac{\pi}{4}(\kappa-1)}}{\sqrt{K}} \sideset{}{'}\sum_{\substack{-K \leq \alpha \leq K \\ \alpha \in 2\mathbb{Z}+1}} \check{q}^{pq^*\alpha^2} \check{q}^{2n\alpha} = \frac{e^{i\frac{\pi}{4}(\kappa-1)}}{\sqrt{K}} \check{q}^{-p^* q n^2} \sideset{}{'}\sum_{\substack{-K \leq \alpha \leq K \\ \alpha \in 2\mathbb{Z}+1}} \check{q}^{pq^*(\alpha + np^* q)^2}. \tag{3.3}$$

Since (for $p^* \in \mathbb{Z}$)

$$\sideset{}{'}\sum_{\substack{-K\le\alpha\le K\\ \alpha\in 2\mathbb{Z}+1}} \check{q}^{pq^*(\alpha+np^*q)^2} = \sideset{}{'}\sum_{\substack{-K\le\alpha\le K\\ \alpha\in 2\mathbb{Z}+1}} \check{q}^{pq^*\alpha^2} = \sum_{\frac{3-K}{2}<\tilde{\alpha}\le\frac{K-1}{2}} \check{q}^{pq^*(2\tilde{\alpha}+1)^2}$$
$$= \sum_{\frac{3-K}{2}<\tilde{\alpha}\le\frac{K-1}{2}} \check{q}^{4pq^*(\tilde{\alpha}+2^*)^2} = \sqrt{K}e^{i\frac{\pi}{4}(1-\kappa)}\left(\frac{pq^*}{K}\right), \tag{3.4}$$

we find that

$$\frac{e^{i\frac{\pi}{4}(\kappa-1)}}{\sqrt{K}} \sideset{}{'}\sum_{\substack{-K\le\alpha\le K\\ \alpha\in 2\mathbb{Z}+1}} \check{q}^{pq^*\alpha^2}\check{q}^{2n\alpha} = \left(\frac{pq^*}{K}\right)\check{q}^{-p^*qn^2}. \tag{3.5}$$

We substitute $\check{q} = 1 + x$ in $\check{q}^{2n\alpha}$ and $\check{q}^{-p^*qn^2}$. After going from $\mathbb{Z}[\check{q}]$ to the factor-ring $\mathbb{Z}'_K[x] = \mathbb{Z}_K[x]/x^{\frac{K+1}{2}}\mathbb{Z}_K[x]$ and using the "checked binomial polynomial"

$$P^\vee_m(\alpha) = (m!)^*\alpha(\alpha-1)\cdots(\alpha-m+1) = (m!)^*\sum_{l=0}^{m} C_{m,l}\alpha^l, \tag{3.6}$$

we find that

$$\left[\sum_{m=0}^{K-1}\sum_{l=0}^{\left[\frac{m}{2}\right]}(m!)^*C_{m,2l}G(p,q;l)4^l n^{2l}x^{m-l}\right]^\diamond = \left(\frac{pq^*}{K}\right)\left[\sum_{m=0}^{\frac{K-1}{2}} x^m\sum_{l=0}^{m}(m!)^*C_{m,l}(-1)^l(p^*q)^l n^{2l}\right]^\diamond \tag{3.7}$$

We limited the sum over m in the *l.h.s.* of this equation to $m \le K-1$ because for $m \ge K$ the minimum power of x^{m-l} is greater that $\frac{K-1}{2}$. Note that since $m \le K-1$, then $(m!)^*$ is well defined.

If we substitute the expansion

$$[G(p,q;l)]^\diamond = \sum_{m=0}^{\frac{K-1}{2}-l} G_m(p,q;l)\,x^m + x^{\frac{K+1}{2}-l}w, \qquad w\in\mathbb{Z}[\check{q}] \tag{3.8}$$

into eq. (3.7), then we can find all the coefficients $G_m(p,q;l)$ by equating the coefficients of *l.h.s.* and *r.h.s.* of eq. (3.7) at equal powers of x and n. These coefficients have to be equal due to the following simple lemma:

Lemma 3.1 *If a degree of a polynomial $P(n)\in\mathbb{Z}_K[n]$ is less than K and $P(n)=0$ for all $n\in\mathbb{Z}_K$, then all the coefficients of $P(n)$ are zero modulo K.*

The proof follows from the fact that the $K \times K$ Van-der-Monde determinant in $\mathbb{Z}_K$ is non-zero.

Each gaussian sum $G(p, q; l)$ appears in the *l.h.s.* of eq. (3.7) with its own power of n: n^{2l}. Therefore the coefficients $G_{m'}(p, q; l)$ of eq. (3.8) can be calculated by "dividing" the polynomial $(-1)^l (p^* q)^l \sum_{m=l}^{\frac{K-1}{2}} (m!)^* C_{m,l} x^m$ appearing at n^{2l} in the *r.h.s.* of eq. (3.7) by the polynomial $\sum_{m=2l}^{K-1} C_{m,2l} 4^l x^{m-l}$ appearing in the *r.h.s.* of that equation at the same power on n. The division is not quite well-defined, hence the indeterminacy in the elements w of eq. (3.8).

This whole calculation of dividing the polynomials can be made more explicit if we go back to eq. (3.5) and make the following substitutions:

$$\check{q}^{2n\alpha} = e^{2n\alpha \log(1+x)} = \sum_{l=0}^{\infty} \frac{(2n\alpha)^l}{l!} (\log(1+x))^l, \tag{3.9}$$

$$\check{q}^{-pq^* n^2} = e^{-pq^* n^2 \log(1+x)} = \sum_{l=0}^{\infty} (-1)^l \frac{(pq^* n^2)^l}{l!} (\log(1+x))^l. \tag{3.10}$$

After "checking" the logarithm

$$\log^\vee(1+x) = x \sum_{n=0}^{K-2} (-1)^n (n+1)^* x^n \tag{3.11}$$

we see that eq. (3.7) transforms into

$$\begin{aligned}\left[\sum_{l=0}^{\frac{K-1}{2}} G(p, q; l)(2n)^{2l}((2l)!)^* (\log^\vee(1+x))^l \left(\frac{\log^\vee(1+x)}{x}\right)^l\right]^\diamond & \\ = \left(\frac{pq^*}{K}\right) \left[\sum_{l=0}^{\frac{K-1}{2}} (-1)^l n^{2l} (pq^*)^l (l!)^* (\log^\vee(1+x))^l\right]^\diamond &.\end{aligned} \tag{3.12}$$

Thus we find that

$$\begin{aligned}[G(p, q; m)]^\diamond &= (-1)^m \left(\frac{pq^*}{K}\right) (2^*)^{2m} (2m)! (m!)^* \left[\left(\frac{x}{\log^\vee(1+x)}\right)^m\right]^\diamond \\ &\quad + x^{\frac{K+1}{2} - m} w, \qquad w \in \mathbb{Z}[\check{q}].\end{aligned} \tag{3.13}$$

Consider now the following identity which is an integral analog of eq. (3.5):

$$\int_{-\infty}^{+\infty} d\alpha \, \check{q}^{\frac{p}{q}\alpha^2} \check{q}^{2n\alpha} = e^{i\frac{\pi}{4}\operatorname{sign}\left(\frac{p}{q}\right)} \left(\frac{K}{2}\right)^{\frac{1}{2}} \left|\frac{q}{p}\right|^{\frac{1}{2}} \check{q}^{-\frac{q}{p}n^2}. \tag{3.14}$$

After substituting eqs. (3.9), (3.10) we find that

$$\int_{-\infty}^{+\infty} d\alpha\, \check{q}^{\frac{p}{q}\alpha^2} \alpha^{2m} x^m = e^{i\frac{\pi}{4}\operatorname{sign}\left(\frac{p}{q}\right)} \left(\frac{K}{2}\right)^{\frac{1}{2}} \left|\frac{q}{p}\right|^{\frac{1}{2}} \frac{(-1)^m (2m)!}{4^m m!} \left(\frac{x}{\log(1+x)}\right)^m \tag{3.15}$$

Eq. (3.2) follows from comparing eq. (3.15) to eq. (3.13).

The formula (3.2) can be generalized to the type of summands that appear in eq. (2.8) after the substitutions (2.9) and (2.12).

Proposition 3.2 *For* $p \neq 0 \pmod{K}$, $0 \leq m \leq K$

$$\left[\frac{e^{i\frac{\pi}{4}(\kappa-1)}}{\sqrt{K}} \sum_{\substack{-K\leq\alpha\leq K \\ \alpha\in 2\mathbb{Z}+1}}{}' \check{q}^{4^* p\alpha^2} P_m\left(\frac{\alpha-1+p^*}{2}\right) x^{\left[\frac{m}{2}\right]} \right]^{\diamond} \tag{3.16}$$

$$= \left(\frac{p}{K}\right) \left[e^{-i\frac{\pi}{4}\operatorname{sign}(q)} \left(\frac{|p|}{2K}\right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} d\alpha\, e^{\frac{i\pi}{2K}p\alpha^2} P_m\left(\frac{\alpha-1+\frac{1}{p}}{2}\right) x^{\left[\frac{m}{2}\right]} \right]^{\diamond} + x^{\frac{K+1}{2}-\left[\frac{m}{2}\right]} w,$$
$$w \in \mathbb{Z}[\check{q}].$$

To prove the proposition for $m < K$ we substitute $P_m^{\vee}(2^*(\alpha-1+p^*))$ for $P_m\left(\frac{\alpha-1+p^*}{2}\right)$ in the *l.h.s.* of this equation and then take the sum for each monomial of $P_m^{\vee}(2^*(\alpha-1+p^*))$ separately. If m is odd then the highest power α^m does not contribute to the sum, therefore the factor $x^{\left[\frac{m}{2}\right]}$ is enough to apply eq. (3.2). We also used the multiplicativity of Legendre symbol and $\left(\frac{4^*}{K}\right) = 1$, since $4^* = (2^*)^2$.

The case of $m = K$ requires a special care. We start with the *l.h.s.* of eq. (3.16). We can use the symmetry of the summation range and gaussian exponent in order to substitute

$$P_K^{(\mathrm{ev})}(\alpha) = \frac{1}{2}\left(P_K\left(\frac{\alpha-1+p^*}{2}\right) + P_K\left(\frac{-\alpha-1+p^*}{2}\right)\right) \tag{3.17}$$

instead of $P_K\left(\frac{\alpha-1+p^*}{2}\right)$. The even polynomial (3.17) takes integer values for odd α and its degree is equal to $K-1$. Therefore the highest divisor of denominators of its coefficients is $K-1$ and we can apply all our previous results to the calculation of the contribution of its monomials. Eq. (3.16) indicates that we need to determine only the terms of order x^0, hence we are interested only in the contribution of the highest degree monomial

$$\frac{p^*-K}{2^K\,(K-1)!}\alpha^{\frac{K-1}{2}}, \tag{3.18}$$

which is determined with the help of eq. (3.13). Consider now the *r.h.s.* of eq. (3.16). Again we substitute the even polynomial

$$P_K^{(\mathrm{ev})}(\alpha)=\frac{1}{2}\left(P_K\left(\frac{\alpha-1+\frac{1}{p}}{2}\right)+P_K\left(\frac{-\alpha-1+\frac{1}{p}}{2}\right)\right). \tag{3.19}$$

Some of its denominators may have K as a divisor, but according to eq. (3.16) we are interested only in the contribution of the highest power of α:

$$\frac{\frac{1}{p}-K}{2^K\,(K-1)!}\alpha^{\frac{K-1}{2}}. \tag{3.20}$$

Comparing it to monomial (3.18) and applying eq. (3.2) to their contributions we arrive at eq. (3.16). This ends the proof of the Proposition 3.2.

Next we move to the polynomials $\tilde{D}_{m,n}$ which participate in the expansion eq. (2.9):

Proposition 3.3 *The gaussian sums and integrals of polynomials $\tilde{D}_{m,n}$ are related by the equation*

$$\begin{aligned}&\left[\frac{e^{i\frac{\pi}{4}N(\kappa-1)}}{K^{\frac{N}{2}}}\sum'_{\substack{-K\le\alpha_j\le K\\ \alpha_j\in 2\mathbb{Z}+1\\ (1\le j\le N)}}\check{q}^{4^*\sum_{j=1}^N p_j\alpha_j^2}\left(\prod_{j=1}^N\left(\alpha_j+p_j^*\right)\right)\sum_{m\le\frac{3}{4}n}\tilde{D}_{m,n}(\alpha_1+p_1^*,\dots,\alpha_N+p_N^*)\,x^n\right]^\diamond\\ &\qquad=\left(\frac{\prod_{j=1}^N p_j}{K}\right)\left[e^{-i\frac{\pi}{4}\sum_{j=1}^N\mathrm{sign}(p_j)}(2K)^{-\frac{N}{2}}\left|\prod_{j=1}^N p_j\right|^{\frac{1}{2}}\right.\\ &\quad\left.\times\int_{-\infty}^{+\infty}d\alpha\,e^{\frac{i\pi}{2K}\sum_{j=1}^N p_j\alpha_j^2}\left(\prod_{j=1}^N\left(\alpha_j+\frac{1}{p_j}\right)\right)\sum_{m\le\frac{3}{4}n}\tilde{D}_{m,n}(\alpha_1+\frac{1}{p_1},\dots,\alpha_N+\frac{1}{p_N})\right]^\diamond.\end{aligned} \tag{3.21}$$

To prove this proposition we rearrange the representation (2.12) in the following form:

$$\begin{aligned}&\left(\prod_{j=1}^N\alpha_j\right)\sum_{m\le\frac{3}{4}n}\tilde{D}_{m,n}(\alpha_1,\dots,\alpha_N)\,x^n\\ &=x^{n-\sum_{j=1}^N[\frac{m_j}{2}]}\sum_{\substack{1\le m_j\le 2\bar{m}_j+1\\ (1\le j\le N)}}C^{(n)}_{m_1,\dots,m_N}x^{[\frac{m_1}{2}]}P_{m_1}\left(\frac{\alpha_1-1}{2}\right)\cdots x^{[\frac{m_N}{2}]}P_{m_N}\left(\frac{\alpha_N-1}{2}\right).\end{aligned} \tag{3.22}$$

We know from Lemma 2.2 that the contribution of each polynomial $x^{[\frac{m_j}{2}]}P_{m_j}\left(\frac{\alpha_j-1}{2}\right)$ to the *l.h.s.* of eq. (3.21) belongs to $\mathbb{Z}[\check{q}]$. Therefore the contribution of the whole expression (3.22)

starts at $x^{n-\sum_{j=1}^{N}\left[\frac{m_j}{2}\right]}$. Hence we may assume that

$$n-\sum_{j=1}^{N}\left[\frac{m_j}{2}\right]\leq\frac{K-1}{2}, \tag{3.23}$$

otherwise the contribution of the polynomial

$$x^{n-\sum_{j=1}^{N}\left[\frac{m_j}{2}\right]}C^{(n)}_{m_1,\ldots,m_N}x^{\left[\frac{m_1}{2}\right]}P_{m_1}\left(\frac{\alpha_1-1}{2}\right)\cdots x^{\left[\frac{m_N}{2}\right]}P_{m_N}\left(\frac{\alpha_N-1}{2}\right). \tag{3.24}$$

is annihilated by the homomorphism $^\diamond$.

The inequalities (2.14) and (3.23) mean that $m_j\leq\frac{K-1}{2}$, so we can apply the Proposition 3.2 to every polynomial $x^{\left[\frac{m_j}{2}\right]}P_{m_j}\left(\frac{\alpha_j-1}{2}\right)$. The terms $x^{\frac{K+1}{2}-\left[\frac{m_j}{2}\right]}w$ of eq. (3.16) can be neglected because

$$\left[x^{n-\sum_{i=1}^{N}\left[\frac{m_i}{2}\right]}\,x^{\frac{K+1}{2}-\left[\frac{m_j}{2}\right]}\,w\right]^\diamond=\left[x^{\frac{K+1}{2}+\left(n-\sum_{i=1}^{N}\left[\frac{m_i}{2}\right]-\left[\frac{m_j}{2}\right]\right)}\,w\right]^\diamond=0 \tag{3.25}$$

in view of the inequality (2.14). This proves the Proposition 3.2.

Now we can prove the Proposition 1.1. We substitute eq. (2.9) into eq. (2.5), apply the homomorphism $^\diamond$ and retain only the relevant terms from the sum of eq. (2.9). Since the contribution of the *l.h.s.* of eq. (3.22) starts at $x^{n-\sum_{j=1}^{N}\left[\frac{m_j}{2}\right]}$ and $\sum_{j=1}^{N}\left[\frac{m_j}{2}\right]\leq m\leq\frac{3}{4}n$, it is enough to retain only the terms with $n\leq 2(K-1)$:

$$\begin{aligned}
[Z'(M;k)]^\diamond \;=\; & \Bigg[e^{i\frac{\pi}{4}(\kappa-1)\sum_{j=1}^{N}(\mathrm{sign}(p_j)-1)}\left(\prod_{j=1}^{N}\mathrm{sign}\,(p_j)\right)\check{q}^{4^*\sum_{j=1}^{N}(3\,\mathrm{sign}(p_j)-p_j-p_j^*)}\,\frac{e^{i\frac{\pi}{4}N(\kappa-1)}}{K^{\frac{N}{2}}}\\
&\times\sum_{\substack{-K\leq\alpha_j\leq K\\ \alpha_j\in 2\mathbb{Z}+1\\ (1\leq j\leq N)}}{}^{'}\;\check{q}^{4^*\sum_{j=1}^{N}p_j\alpha_j^2}\left(\prod_{j=1}^{N}\left(\alpha_j+p_j^*\right)\right)\sum_{n=0}^{2(K-1)}\sum_{m\leq\frac{3}{4}n}\tilde{D}_{m,n}(\alpha_1+p_1^*,\ldots,\alpha_N+p_N^*)x^n\Bigg]^\diamond\\
=\; & \left(\frac{\prod_{j=1}^{N}p_j}{K}\right)e^{i\frac{\pi}{4}(\kappa-1)\sum_{j=1}^{N}(\mathrm{sign}(p_j)-1)}\\
&\times\Bigg[e^{-i\frac{\pi}{4}\sum_{j=1}^{N}\mathrm{sign}(p_j)}\left(\prod_{j=1}^{N}\mathrm{sign}\,(p_j)\right)\check{q}^{\frac{1}{4}\sum_{j=1}^{N}(3\,\mathrm{sign}(p_j)-p_j-p_j^*)}\,\frac{\left|\prod_{j=1}^{N}p_j\right|^{\frac{1}{2}}}{(2K)^{\frac{N}{2}}}\\
&\times\int_{-\infty}^{+\infty}d\alpha_1\cdots d\alpha_N\,e^{\frac{i\pi}{2K}\sum_{j=1}^{N}p_j\alpha_j^2}\\
&\times\left(\prod_{j=1}^{N}\left(\alpha_j+\frac{1}{p_j}\right)\right)\sum_{n=0}^{2(K-1)}\sum_{m\leq\frac{3}{4}n}\tilde{D}_{m,n}(\alpha_1+\frac{1}{p_1},\ldots,\alpha_N+\frac{1}{p_N})x^n\Bigg]^\vee.
\end{aligned} \tag{3.26}$$

Here we used an identity

$$\left[\check{q}^{4^*\sum_{j=1}^{N}(3\,\mathrm{sign}(p_j)-p_j-p_j^*)}\right]^{\diamond}=\left[\check{q}^{\frac{1}{4}\sum_{j=1}^{N}(3\,\mathrm{sign}(p_j)-p_j-p_j^*)}\right]^{\vee}. \tag{3.27}$$

We can extend the sum over n in the *r.h.s.* of eq. (3.27) to all $n \geq 0$ because the contribution of $\tilde{D}_{m,n}x^n$ with $n > 2(K-1)$ starts above $x^{\frac{K-1}{2}}$. As a result, we obtain the full Jones polynomial. Now using the identities

$$e^{i\frac{\pi}{4}(\kappa-1)(\mathrm{sign}(p)-1)}\left(\frac{p}{K}\right)=\left(\frac{|p|}{K}\right), \tag{3.28}$$

$$\left|\prod_{j=1}^{N}p_j\right|=|H_1(M,\mathbb{Z})| \tag{3.29}$$

$$\mathrm{sign}\,(p)=ie^{-i\frac{\pi}{2}\,\mathrm{sign}(p)}, \tag{3.30}$$

we get the following formula

$$\begin{aligned}[Z'(M;k)]^{\diamond} = & \left(\frac{|H_1(M,\mathbb{Z})|}{K}\right)\left[e^{-\frac{3}{4}i\pi\sum_{j=1}^{N}\mathrm{sign}(p_j)}\frac{i^N}{(2K)^{\frac{N}{2}}}e^{\frac{i\pi}{2K}\sum_{j=1}^{N}\left(\mathrm{sign}(p_j)-p_j-\frac{1}{p_j}\right)}\left|\prod_{j=1}^{N}p_j\right|^{\frac{1}{2}}\right. \\ & \left.\times\int\limits_{\substack{-\infty\\ [\alpha_j=0]}}^{+\infty}d\alpha_1\cdots d\alpha_N\, e^{\frac{i\pi}{2K}\sum_{j=1}^{N}p_j\alpha_j^2}J_{\alpha_1+\frac{1}{p_1},\ldots,\alpha_N+\frac{1}{p_N}}(\mathcal{L};k)\right]^{\vee}\end{aligned} \tag{3.31}$$

Combining it with eqs. (1.27) and (1.18) and using eq. (3.29) again we easily arrive at eq. (1.23). Note that eq. (3.29) guarantees that $p_j \neq 0 \pmod K$ if $|H_1(M,\mathbb{Z})| \neq 0 \pmod K$. This proves the Proposition 1.1.

4 A General Rational Surgery Formula

Up until this point we were working only with surgeries of the type $(p,1)$. This was enough to prove the theorems of Section 1, because H. Murakami and T. Ohtsuki showed [6] that any RHS M can be constructed by $(p_j,1)$ surgeries on an ASL in S^3 up to a connected sum of lens spaces $L_{p'_j,1}$, that is, instead of M one might end up with $M\#L_{p'_1,1}\#\ldots L_{p'_n,1}$. However from the technical point of view it would be better to have a formula for the invariant

$Z'(M;k)$ of a manifold constructed by general rational surgeries (p_j,q_j) on the components of an N-component link $\mathcal{L}$ in S^3.

The formula for Witten's invariant $Z(M;k)$ was derived by L. Jeffrey [10]

$$
\begin{aligned}
Z(M;k) \;=\; & Z(S^3;k)\exp\left[i\frac{\pi}{4}\frac{K-2}{K}\left(\sum_{j=1}^{N}\Phi(U^{(p_j,q_j)}-3\,\mathrm{sign}\,(L)\right)\right] \\
& \times \sum_{1\le\alpha_1,\dots,\alpha_N\le K-1} J_{\alpha_1,\dots,\alpha_N}(\mathcal{L};k)\prod_{j=1}^{N}\tilde{U}^{(p_j,q_j)}_{\alpha_j 1},
\end{aligned}
\tag{4.1}
$$

here L is the linking matrix of $\mathcal{L}$, $\frac{p_j}{q_j}$ being the self-linking numbers. The matrices

$$
U^{(p_j,q_j)} = \begin{pmatrix} p_j & r_j \\ q_j & s_j \end{pmatrix} \in SL(2,\mathbb{Z})
\tag{4.2}
$$

describe the surgeries (a meridian on the tubular neighborhood is glued to p_j(meridian) + q_j(parallel) of the link complement),

$$
\Phi\begin{bmatrix} p & r \\ q & s \end{bmatrix} = \frac{p+s}{q} - 12s(p,q),
\tag{4.3}
$$

$s(p,q)$ being the Dedekind sum, and

$$
\begin{aligned}
\tilde{U}^{(p,q)}_{\alpha\beta} \;=\; & i\frac{\mathrm{sign}\,(q)}{\sqrt{2K|q|}}e^{-i\frac{\pi}{4}\Phi(U^{(p,q)})}\sum_{n=0}^{q-1}\sum_{\mu=\pm1}\mu \\
& \times\exp\left[\frac{i\pi}{2Kq}(p\alpha^2-2\alpha(2Kn+\mu\beta)+s(2Kn+\mu\beta)^2)\right].
\end{aligned}
\tag{4.4}
$$

Let us introduce some notations. A rational (p_j,q_j) surgery on $\mathcal{L}_j$ can be presented as a combination of $(m_t^{(j)},1)$, $1\le t\le \bar{t}^{(j)}$ surgeries on a chain of unknots simply linked to $\mathcal{L}_j$ (see *e.g.* [9], [10] and references therein) such that

$$
U^{(p_j,q_j)} = T^{m^{(j)}_{\bar{t}^{(j)}}}ST^{m^{(j)}_{\bar{t}^{(j)}-1}}S\cdots T^{m^{(j)}_1}S,
\tag{4.5}
$$

here

$$
S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\tag{4.6}
$$

We denote this chain (including $\mathcal{L}_j$ itself) as $\tilde{\mathcal{L}}_j$ and all the chains of $\mathcal{L}$ as $\tilde{\mathcal{L}}$. For $1 \le t \le \bar{t}^{(j)}$ we set

$$U^{(p_t^{(j)}, q_t^{(j)})} = T^{m_t^{(j)}} S \cdots T^{m_1^{(j)}} S, \tag{4.7}$$

so that $p_{\bar{t}^{(j)}}^{(j)} = p_j$, $q_{\bar{t}^{(j)}}^{(j)} = q_j$. From now on we assume for simplicity that none of the numbers $q_t^{(j)}$ is divisible by K. Then we are going to prove the following:

Proposition 4.1 *Let M be a manifold constructed by (p_j, q_j) surgeries on an N-component link $\mathcal{L}$ in S^3. Then*

$$\begin{aligned} Z'(M;k) \;=\; & \left(\frac{\left|\prod_{j=1}^N q_j\right|}{K}\right) \frac{\prod_{j=1}^N \operatorname{sign}(q_j)}{K^{\frac{N}{2}}} e^{-i\frac{\pi}{4}\kappa \operatorname{sign}(L)} e^{-i\pi\frac{3}{4}\frac{K-2}{K}\operatorname{sign}(L)} \\ & \times \check{q}^{-4^* \sum_{j=1}^N \Phi(U^{(p_j,q_j)})} \check{q}^{\left(2^* - \frac{1}{2}\right)\sum_{j=1}^N \operatorname{sign}\left(\frac{p_j}{q_j}\right)} \\ & \times \sideset{}{'}\sum_{\substack{-K \le \alpha_j \le K \\ \alpha_j \in 2\mathbb{Z}+1 \\ (1 \le j \le N)}} J_{\alpha_1,\ldots,\alpha_N}(\mathcal{L};k)\, \check{q}^{4^*\sum_{j=1}^N q_j^*(p_j\alpha_j^2 + s_j)} \prod_{j=1}^N \left(\frac{i}{2}\sum_{\mu_j = \pm 1} \mu_j \check{q}^{-2^* q_j^* \mu_j \alpha_j}\right) \end{aligned} \tag{4.8}$$

We could use the general surgery formula (4.8) instead of the $(p_j, 1)$ surgery formula (1.6) throughout the Sections 2 and 3 in order to produce a somewhat more flexible proof of Theorems 1.1, 1.2 and Proposition 1.1.

We begin the proof of the Proposition 4.1 by recalling the Kirby-Melvin formula [3] which expresses $Z'(M;k)$ in terms of data associated to $\tilde{\mathcal{L}}$:

$$\begin{aligned} Z'(M;k) \;=\; & e^{i\frac{\pi}{4}\frac{K-2}{K}\left[\sum_{j=1}^N \Phi(U^{(p_j,q_j)}) - 3\operatorname{sign}(L)\right]} e^{-i\frac{\pi}{4}\kappa \operatorname{sign}\left(\tilde{L}\right)} \\ & \times \sideset{}{'}\sum_{\substack{-K \le \alpha_j \le K \\ \alpha_j \in 2\mathbb{Z}+1 \\ (1 \le j \le N)}} J_{\alpha_1,\ldots,\alpha_N}(\mathcal{L};k) \prod_{j=1}^N \check{U}_{\alpha_j 1}^{(p_j,q_j)}, \end{aligned} \tag{4.9}$$

here (we drop the index j in eq. (4.5))

$$\check{U}_{\alpha\beta}^{(p,q)} = \left(\check{T}^{m_{\bar{t}}} \check{S}\, \check{T}^{m_{\bar{t}-1}} \check{S} \cdots \check{T}^{m_1} \check{S}\right)_{\alpha\beta}, \tag{4.10}$$

$$\check{T}_{\alpha\beta} = e^{-i\frac{\pi}{4}} \check{q}^{\frac{1}{4}\alpha^2} \delta_{\alpha\beta}, \qquad \check{S}_{\alpha\beta} = \frac{1}{\sqrt{K}} \sin\left(\frac{\pi}{K}\alpha\beta\right) \tag{4.11}$$

and $\tilde{L}$ is the linking matrix of the "expanded" link $\tilde{\mathcal{L}}$.

The following lemma presents an explicit expression for $\check{U}^{(p,q)}_{\alpha\beta}$, which is similar to that of eq. (4.4):

Lemma 4.1

$$\check{U}^{(p,p)}_{\alpha\beta} = \left(\frac{|q|}{K}\right) \frac{\mathrm{sign}\,(q)}{\sqrt{K}} e^{-i\frac{\pi}{4}\Phi(U^{(p,q)})}\, e^{-i\frac{\pi}{4}\kappa \sum_{t=0}^{\bar{t}-1} \mathrm{sign}\left(\frac{p_t}{q_t}\right)} \tag{4.12}$$
$$\times \check{q}^{\left(\frac{1}{4}-4^*\right)\sum_{t=1}^{\bar{t}} m_t + \left(2^*-\frac{1}{2}\right)\sum_{t=1}^{\bar{t}} \mathrm{sign}\left(\frac{p_t}{q_t}\right)}\, \check{q}^{4^*q^*(p\alpha^2+s\beta^2)}\, \frac{i}{2} \sum_{\mu=\pm 1} \mu\, \check{q}^{-2^*q^*\mu\alpha\beta}.$$

To prove the lemma we slightly change eq. (4.11):

$$\check{T}_{\alpha\beta} = e^{-i\frac{\pi}{4}}\, \check{q}^{\frac{1}{4}-4^*}\, \check{q}^{4^*\alpha^2}\, \delta_{\alpha\beta}, \qquad \check{S}_{\alpha\beta} = \frac{1}{\sqrt{K}} \frac{i}{2}\, \check{q}^{\pm\left(2^*-\frac{1}{2}\right)} \sum_{\mu=\pm 1} \mu\, \check{q}^{-2^*\mu\alpha\beta}, \tag{4.13}$$

the choice of sign in $\check{q}^{\pm\left(2^*-\frac{1}{2}\right)}$ is arbitrary.

We prove eq. (4.12) by induction on $\bar{t}$. If $\bar{t}=1$, that is, if $U^{(p,q)} = T^m S$, then the check is trivial if we recall that $\Phi(T^m S) = m$. It is also easy to check eq. (4.12) for $U^{(p+mq,q)} = T^m U^{(p,q)}$. It remains to check eq. (4.12) for $U^{(-q,p)} = SU^{(p,q)}$.

We have to calculate the sum:

$$\check{U}^{(-q,p)}_{\alpha\beta} = \sideset{}{'}\sum_{\substack{-K\le\gamma\le K \\ \gamma\in 2\mathbb{Z}+1}} \check{S}_{\alpha\gamma} \check{U}^{(p,q)}_{\gamma\beta}. \tag{4.14}$$

The following gaussian sum is at the center of this calculation:

$$\begin{aligned} &\sideset{}{'}\sum_{\substack{-K\le\gamma\le K \\ \gamma\in 2\mathbb{Z}+1}} \sum_{\mu_{1,2}=\pm 1} \mu_1\mu_2\, \check{q}^{4^*q^*p\gamma^2 - 2^*\gamma(q^*\mu_1\beta+\mu_2\alpha)+4^*q^*s\beta^2} \\ &\quad = \sideset{}{'}\sum_{\substack{-K\le\gamma\le K \\ \gamma\in 2\mathbb{Z}+1}} \sum_{\mu_{1,2}=\pm 1} \mu_1\mu_2\, \check{q}^{4^*q^*p(\gamma-p^*\mu_1\beta-p^*q\mu_2\alpha)^2 - 4^*q^*p^*(\mu_1\beta+q\mu_2\alpha)^2+4^*q^*s\beta^2} \\ &\quad = 2\sqrt{K} e^{i\frac{\pi}{4}(\kappa-1)} \left(\frac{pq^*}{K}\right) \check{q}^{4^*p^*(-q\alpha^2+r\beta^2)} \sum_{\mu=\pm 1} \mu\, \check{q}^{-2^*p^*\mu\alpha\beta}, \end{aligned} \tag{4.15}$$

here we used the following relations:

$$\sideset{}{'}\sum_{\substack{-K\le\gamma\le K \\ \gamma\in 2\mathbb{Z}+1}} \check{q}^{4^*pq^*\gamma^2} = e^{i\frac{\pi}{4}(\kappa-1)} \sqrt{K} \left(\frac{pq^*}{K}\right), \tag{4.16}$$

$$s - p^* = p^*qr, \tag{4.17}$$

the latter relation follows from $ps - qr = 1$. To complete the verification of eq. (4.12) we recall the following identities:

$$\Phi(SU^{(p,q)}) = \Phi(U^{(p,q)}) - 3\,\mathrm{sign}\left(\frac{p}{q}\right), \tag{4.18}$$

$$i\,\mathrm{sign}\,(q) = e^{i\frac{\pi}{2}\,\mathrm{sign}\left(\frac{p}{q}\right)}\,\mathrm{sign}\,(p)\,, \tag{4.19}$$

$$e^{i\frac{\pi}{4}(\kappa-1)}\left(\frac{|q|}{K}\right)\left(\frac{pq^*}{K}\right) = e^{i\frac{\pi}{4}(\kappa-1)\,\mathrm{sign}\left(\frac{p}{q}\right)}\left(\frac{|p|}{K}\right). \tag{4.20}$$

This ends the proof of the lemma.

To finish the proof of Proposition 4.1 we rearrange some phase factors of eqs. (4.9) and (4.12). We substitute a relation

$$\mathrm{sign}\left(\tilde{L}\right) = \mathrm{sign}\,(L) + \sum_{j=1}^{N}\sum_{t=1}^{t^{(j)}-1}\mathrm{sign}\left(\frac{p_t^{(j)}}{q_t^{(j)}}\right) \tag{4.21}$$

into the factor $e^{-i\frac{\pi}{4}\kappa\,\mathrm{sign}\left(\tilde{L}\right)}$ of eq. (4.9). Then we calculate the combination of phases coming from that factor and from eq. (4.12) (we drop the index j):

$$\begin{aligned}
&\check{q}^{\left(\frac{1}{4}-4^*\right)\sum_{t=1}^{\bar{t}} m_t+\left(2^*-\frac{1}{2}\right)\sum_{t=1}^{\bar{t}}\mathrm{sign}\left(\frac{p_t}{q_t}\right)}\; e^{-i\frac{\pi}{2}\kappa\sum_{t=1}^{\bar{t}-1}\mathrm{sign}\left(\frac{p_t}{q_t}\right)} \\
&\quad = \check{q}^{\left(\frac{1}{4}-4^*\right)\sum_{t=1}^{\bar{t}} m_t+\left(2^*-\frac{1}{2}\right)\mathrm{sign}\left(\frac{p}{q}\right)}\;\check{q}^{\left(2^*-\frac{1}{2}-\frac{\kappa K}{4}\right)\sum_{t=1}^{\bar{t}-1}\mathrm{sign}\left(\frac{p_t}{q_t}\right)} \\
&\quad = \check{q}^{\left(\frac{1}{4}-4^*\right)\left[\sum_{t=1}^{\bar{t}} m_t-3\sum_{t=1}^{\bar{t}-1}\mathrm{sign}\left(\frac{p_t}{q_t}\right)\right]}\;\check{q}^{\left(2^*-\frac{1}{2}\right)\mathrm{sign}\left(\frac{p}{q}\right)} \\
&\quad = \check{q}^{\left(\frac{1}{4}-4^*\right)\Phi(U^{(p,q)})}\;\check{q}^{\left(2^*-\frac{1}{2}\right)\mathrm{sign}\left(\frac{p}{q}\right)}
\end{aligned} \tag{4.22}$$

We used here eq. (2.7) and the formula

$$\Phi(U^{(p,q)}) = \sum_{t=1}^{\bar{t}} m_t - 3\sum_{t=1}^{\bar{t}-1}\mathrm{sign}\left(\frac{p_t}{q_t}\right) \tag{4.23}$$

(see [10] and references therein).

A combination of eqs. (4.9), (4.12), (4.21) and (4.22) leads to eq. (4.8). This concludes the proof of Proposition 4.1.

As an application of the Proposition 4.1 let us calculate the invariant of a lens space $L_{p,q}$. This manifold is constructed by a $U^{(-p,q)}$ surgery on an unknot in S^3. Since

$$J_\alpha(\mathrm{unknot};k) = \frac{\sin\left(\frac{\pi}{K}\alpha\right)}{\sin\left(\frac{\pi}{K}\right)}, \tag{4.24}$$

we can say that $L_{p,q}$ is constructed by a chain surgery $U^{(-q,-p)} = SU^{(-p,q)}$ applied to an empty knot, times a factor $\frac{\sqrt{K}}{\sin\left(\frac{\pi}{K}\right)}$. Then we can read the result directly from eq. (4.8) by setting there $N=1$, $\mathrm{sign}\,(L)=0$, $\alpha=1$ and

$$U^{(-q,-p)} = \begin{pmatrix} -q & s \\ -p & r \end{pmatrix} \tag{4.25}$$

instead of $U^{(p,q)}$:

$$\begin{aligned} Z'(L_{p,q}) &= \left(\frac{|p|}{K}\right) \mathrm{sign}\,(p)\ \check{q}^{4^*(p^*(q-r)-\Phi(U^{(-q,-p)}))}\ \check{q}^{2^*}\ \frac{\check{q}^{2^*p^*}-\check{q}^{-2^*p^*}}{\check{q}-1} \\ &= \left(\frac{|p|}{K}\right) \mathrm{sign}\,(p)\ \check{q}^{3s^\vee(q,p)}\ \check{q}^{2^*}\ \frac{\check{q}^{2^*p^*}-\check{q}^{-2^*p^*}}{\check{q}-1} \\ &= \left(\frac{|p|}{K}\right) \mathrm{sign}\,(p)\ \check{q}^{3s^\vee(q,p)}\ (-1)^{p^*+1}\frac{\check{q}^{\frac{p^*}{2}}-\check{q}^{-\frac{p^*}{2}}}{\check{q}^{\frac{1}{2}}-\check{q}^{-\frac{1}{2}}}, \end{aligned} \tag{4.26}$$

here $s^\vee(q,p)$ is the "checked" Dedekind sum, that is, its denominator is inverted modulo K as in eq. (1.13). We used the relation $\check{q}^{2^*} = -\check{q}^{\frac{1}{2}}$ in order to derive the last expression in this equation. Although it might look simpler than the previous one, it obscures the fact that $Z'(L_{p,q}) \in \mathbb{Z}[\check{q}]$. Also note that the expression $\check{q}^{\frac{p^*}{2}}$ by itself is ambiguous since p^* is defined only modulo K.

Comparing the second expression in the *r.h.s.* of eq. (4.26) with the formula for the trivial connection contribution to Witten's invariant of the lens space

$$Z^{(\mathrm{tr})}(L_{p,q};k) = Z(S^3;k)\frac{\mathrm{sign}\,(p)}{\sqrt{|p|}}\check{q}^{3s(p,q)}\frac{\check{q}^{\frac{p}{2}}-\check{q}^{-\frac{p}{2}}}{\check{q}^{\frac{1}{2}}-\check{q}^{-\frac{1}{2}}} \tag{4.27}$$

derived in [10] we can easily check the relation (1.20).

The formula (4.8) also allows us to check the Proposition 1.1 for Seifert manifolds which are rational homology spheres. Consider an $(N+1)$-component link $\mathcal{L}$ in S^3 consisting of N unknots $\mathcal{L}_j$, $1 \le j \le N$ simply linked to a single unknot $\mathcal{L}_0$. A Seifert manifold $X\left(\frac{p_1}{q_1},\ldots,\frac{p_n}{q_n}\right)$ is constructed by performing (p_j,q_j) surgeries on the components $\mathcal{L}_j$ and a $(0,1)$ surgery on $\mathcal{L}_0$ (see, *e.g.* [9]). The Jones polynomial of $\mathcal{L}$ is known [1] to be equal to

$$J_{\beta,\alpha_1,\ldots,\alpha_N}(\mathcal{L};k) = \frac{1}{\sin\left(\frac{\pi}{K}\right)}\frac{\prod_{j=1}^N \sin\left(\frac{\pi}{K}\beta\alpha_j\right)}{\sin^{N-1}\left(\frac{\pi}{K}\beta\right)}, \tag{4.28}$$

here β is the color of $\mathcal{L}_0$ and α_j are the colors of $\mathcal{L}_j$. The signature of $\mathcal{L}$ is equal to

$$\operatorname{sign}(L) = -\operatorname{sign}\left(\frac{H}{P}\right) + \sum_{j=1}^{N} \operatorname{sign}\left(\frac{q_j}{p_j}\right) \tag{4.29}$$

Here we introduced notations

$$P = \prod_{j=1}^{N} p_j, \qquad H = P \sum_{j=1}^{N} \frac{q_j}{p_j}, \tag{4.30}$$

so that

$$\left| H_1\left(X\left(\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}\right)\right)\right| = |H|. \tag{4.31}$$

The Proposition 4.1 provides the following expression for the invariant of the Seifert manifold $X\left(\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}\right)$:

$$\begin{aligned} Z'(X;k) \; = \; & \frac{i}{2\sqrt{K}} e^{i\frac{\pi}{4}\kappa \operatorname{sign}\left(\frac{H}{P}\right)} e^{i\pi\frac{3}{4}\frac{K-2}{K}\operatorname{sign}\left(\frac{H}{P}\right)} \check{q}^{\left(\frac{1}{2}-2^*\right)\operatorname{sign}\left(\frac{H}{P}\right)} \\ & \times \sum_{\substack{-K\le\beta\le K \\ \beta\in 2\mathbb{Z}+1}}{}' (\check{q}^{-2^*\beta} - \check{q}^{2^*\beta}) Z'_\beta(L^{(\mathrm{con})};k), \end{aligned} \tag{4.32}$$

here

$$Z'_\beta(L^{(\mathrm{con})};K) \; = \; \left(\frac{\sin\left(\frac{\pi}{K}\right)}{\sin\left(\frac{\pi}{K}\beta\right)}\right)^{N-1} \prod_{j=1}^{N} Z'_\beta(L_{-p_j,q_j};K), \tag{4.33}$$

$$\begin{aligned} Z'_\beta(L_{-p_j,q_j};K) \; = \; & \left(\frac{|q_j|}{K}\right) \frac{\operatorname{sign}(q_j)}{\sqrt{K}} e^{-i\frac{\pi}{K}\kappa\operatorname{sign}\left(\frac{q_j}{p_j}\right)} e^{-i\pi\frac{3}{4}\frac{K-2}{K}\operatorname{sign}\left(\frac{q_j}{p_j}\right)} \\ & \times \check{q}^{-4^*\Phi(U^{(p_j,q_j)})} \check{q}^{\left(2^*-\frac{1}{2}\right)\operatorname{sign}\left(\frac{q_j}{p_j}\right)} \\ & \times \sum_{\substack{-K\le\alpha_j\le K \\ \alpha_j\in 2\mathbb{Z}+1 \\ (1\le j\le N)}}{}' \frac{\sin\left(\frac{\pi}{K}\beta\alpha_j\right)}{\sin\left(\frac{\pi}{K}\right)} \check{q}^{4^*q_j^*(p_j\alpha_j^2+s_j)} \frac{i}{2}\left(\check{q}^{-2^*q_j^*\alpha_j} - \check{q}^{2^*q_j^*\alpha_j}\right). \end{aligned} \tag{4.34, 4.35}$$

In these formulas $Z'_\beta(L_{-p_j,q_j};k)$ is an invariant of the β-colored link $\mathcal{L}_0$ inside the lens space L_{-p_j,q_j} constructed by the (p_j,q_j) surgery on the unknot $\mathcal{L}_j$. $Z'_\beta(L^{(\mathrm{con})};k)$ is the invariant of the β-colored knot $\mathcal{L}_0$ inside the connected sum of lens spaces

$$L^{(\mathrm{con})} = L_{-p_1,q_1} \# \cdots \# L_{-p_N,q_N}. \tag{4.36}$$

The calculation of invariants $Z'_\beta(L_{-p_j,q_j};k)$ runs similar to that of $Z'(L_{p,q};k)$ of eq. (4.26). The only difference is that by taking a sum over α_j we go from $\tilde{U}^{(p_j,q_j)}_{\alpha_j 1}$ to $\tilde{U}^{(-q_j,p_j)}_{\beta 1}$ rather than to $\tilde{U}^{(-q_j,p_j)}_{11}$. As a result,

$$Z'_\beta(L_{-p_j,q_j};k) = \left(\frac{|p_j|}{K}\right) \mathrm{sign}\,(p_j)\, \check{q}^{4^* p_j^* q_j - 3 s^\vee(q_j,p_j)} \check{q}^{-4^* p_j^* q_j \beta^2} \frac{\check{q}^{-2^* p_j^* \beta} - \check{q}^{2^* p_j^* \beta}}{\check{q}^{-2^*} - \check{q}^{2^*}} \tag{4.37}$$

and

$$\begin{aligned} Z'(X;k) = & \frac{i}{2\sqrt{K}} e^{i\frac{\pi}{4}\kappa\,\mathrm{sign}\left(\frac{H}{P}\right)} e^{i\pi\frac{3}{4}\frac{K-2}{K}\,\mathrm{sign}\left(\frac{H}{P}\right)} \check{q}^{\left(\frac{1}{2}-2^*\right)\mathrm{sign}\left(\frac{H}{P}\right)} \\ & \times \left(\frac{|p|}{K}\right) \mathrm{sign}\,(P)\, \check{q}^{4^* P^* H - 3\sum_{j=1}^N s^\vee(q_j,p_j)} \\ & \times \frac{1}{\check{q}^{-2^*} - \check{q}^{2^*}} \sum_{\substack{-K\le\beta\le K \\ \beta\in 2\mathbb{Z}+1}}' \check{q}^{-4^* P^* H \beta^2} \frac{\prod_{j=1}^N (\check{q}^{-2^* p_j^* \beta} - \check{q}^{2^* p_j^* \beta})}{\check{q}^{-2^*} - \check{q}^{2^*}} \end{aligned} \tag{4.38}$$

The preexponential factor of eq. (4.38) can be put in the form

$$J(\beta)(\check{q}^{-2^*\beta} - \check{q}^{2^*\beta}), \tag{4.39}$$

where

$$J(\beta) = \frac{\prod_{j=1}^N (\check{q}^{-2^* p_j^* \beta} - \check{q}^{2^* p_j^* \beta})}{(\check{q}^{-2^*} - \check{q}^{2^*})(\check{q}^{-2^*\beta} - \check{q}^{2^*\beta})^{N-1}}. \tag{4.40}$$

It is easy to check that the function $J(\beta)$ belongs to $\mathbb{Z}[\check{q}]$ and satisfies the properties of the Jones polynomial described in Proposition 1.2. Therefore the full machinery of Section 3 could be applied to the sum of eq. (4.38) in order to convert it to the integral and ultimately prove the Proposition 1.1 for Seifert manifolds. However there is an easier way. The numbers p_j^*, $1\le j\le N$ determine a set of positive integer numbers $\Lambda(p_1^*,\ldots,p_N^*)\in\mathbb{N}$ and multiplicity factors $C_n(p_1^*,\ldots,p_N^*)\in\mathbb{Z}$, $n\in\Lambda(p_1^*,\ldots,p_N^*)$ such that

$$\frac{\prod_{j=1}^N (\check{q}^{-2^* p_j^* \beta} - \check{q}^{2^* p_j^* \beta})}{(\check{q}^{-2^*\beta} - \check{q}^{2^*\beta})^{N-1}} = \sum_{n\in\Lambda(p_1^*,\ldots,p_N^*)} C_n(p_1^*,\ldots,p_N^*)(\check{q}^{-2^* n\beta} - \check{q}^{2^* n\beta}). \tag{4.41}$$

Now we can apply eq. (3.5) to the calculation of the sum

$$\frac{1}{\check{q}^{-2^*} - \check{q}^{2^*}} \sum_{\substack{-K\le\beta\le K \\ \beta\in 2\mathbb{Z}+1}}' \check{q}^{-4^* P^* H \beta^2} \frac{\prod_{j=1}^N (\check{q}^{-2^* p_j^* \beta} - \check{q}^{2^* p_j^* \beta})}{(\check{q}^{-2^*\beta} - \check{q}^{2^*\beta})^{N-2}} \tag{4.42}$$

$$
\begin{aligned}
&= \frac{2}{\check{q}^{-2^*}-\check{q}^{2^*}} \sum_{n\in\Lambda} C_n \sideset{}{'}\sum_{\substack{-K\le\beta\le K\\ \beta\in 2\mathbb{Z}+1}} \check{q}^{-4^*P^*H\beta^2}\big(\check{q}^{-2^*(n+1)\beta}-\check{q}^{-2^*(n-1)\beta}\big)\\
&= 2\sqrt{K}\left(\frac{P^*H}{K}\right) e^{i\frac{\pi}{4}(\kappa-1)} \sum_{n\in\Lambda} C_n \frac{\check{q}^{4^*PH^*(n+1)^2}-\check{q}^{4^*PH^*(n-1)^2}}{\check{q}^{-2^*}-\check{q}^{2^*}}\\
&= 2\sqrt{K}\left(\frac{P^*H}{K}\right) e^{i\frac{\pi}{4}(\kappa-1)} \sum_{n\in\Lambda} C_n \check{q}^{4^*PH^*(n^2+1)} \frac{\check{q}^{2^*PH^*n}-\check{q}^{-2^*PH^*n}}{\check{q}^{-2^*}-\check{q}^{2^*}}.
\end{aligned}
$$

Combining eqs. (4.38) and (4.42) we find the formula

$$
\begin{aligned}
Z'(X;k) \;=\; &\left(\frac{|H|}{K}\right) \mathrm{sign}\,(H)\, \check{q}^{4^*P^*H-3\cdot 4^*\,\mathrm{sign}\left(\frac{H}{P}\right)-3\sum_{j=1}^{N} s^\vee(q_j,p_j)} \\
&\times \sum_{n\in\Lambda} C_n q^{4^*PH^*(n^2+1)} \frac{\check{q}^{-2^*PH^*n}-\check{q}^{2^*PH^*n}}{\check{q}^{-2^*}-\check{q}^{2^*}},
\end{aligned}
\tag{4.43}
$$

which demonstrates that $Z'(X;k)\in\mathbb{Z}[\check{q}]$.

Now we come back to eqs. (4.41), (4.42) and use the fact that for $n\in\mathbb{Z}$,

$$
\sideset{}{'}\sum_{\substack{-K\le\beta\le K\\ \beta\in 2\mathbb{Z}+1}} \check{q}^{-4^*P^*H\beta^2-2^*n\beta} \;=\; \sqrt{K}\left(\frac{P^*H}{K}\right) e^{i\frac{\pi}{4}(\kappa-1)} \check{q}^{4^*PH^*n^2}, \tag{4.44}
$$

$$
\int_{-\infty}^{+\infty} d\beta\, \check{q}^{-\frac{\beta^2}{4H^*P}-2^*\beta n} \;=\; e^{-i\frac{\pi}{4}\,\mathrm{sign}(PH^*)}\sqrt{2K|PH^*|}\,\check{q}^{4^*PH^*n^2}, \tag{4.45}
$$

so that

$$
\sideset{}{'}\sum_{\substack{-K\le\beta\le K\\ \beta\in 2\mathbb{Z}+1}} \check{q}^{-4^*P^*H\beta^2-2^*n\beta} = \left(\frac{P^*H}{K}\right) e^{i\frac{\pi}{4}(\kappa-1)} \frac{e^{i\frac{\pi}{4}\,\mathrm{sign}(PH^*)}}{\sqrt{2|PH^*|}} \int_{-\infty}^{+\infty} d\beta\, \check{q}^{-\frac{\beta^2}{4H^*P}-2^*\beta n}. \tag{4.46}
$$

This equation allows us to convert the sum over β in eq. (4.42) into an integral. Then by using eq. (4.41) backwards we arrive at eq. (4.38) with the integral instead of a sum:

$$
Z'(X;k) \;=\; \left(\frac{|H|}{K}\right) \mathrm{sign}\,(H)\, \check{q}^{4^*P^*H-3\cdot 4^*\,\mathrm{sign}\left(\frac{H}{P}\right)-3\sum_{j=1}^{N} s^\vee(q_j,p_j)} I(\check{q}), \tag{4.47}
$$

$$
I(\check{q}) \;=\; \frac{1}{\check{q}^{2^*}-\check{q}^{-2^*}} \frac{e^{i\frac{\pi}{4}\,\mathrm{sign}(PH^*)}}{2\sqrt{2K|PH^*|}} \int_{-\infty}^{+\infty} d\beta\, \check{q}^{-\frac{\beta^2}{4PH^*}} \frac{\prod_{j=1}^{N}\big(\check{q}^{-2^*p_j^*\beta}-\check{q}^{2^*p_j^*\beta}\big)}{\big(\check{q}^{-2^*\beta}-\check{q}^{2^*\beta}\big)^{N-2}}. \tag{4.48}
$$

The integral over β is well defined in view of eq. (4.43) (actually, one might add a regularizing factor $\lim_{\epsilon\to 0} e^{-\epsilon\beta^2}$. It can also be calculated by expanding the preexponential factor in powers of $x=q-1$ and integrating their coefficients, which are polynomials in β, with the

gaussian exponential $\check{q}^{-\frac{\beta^2}{4PH^*}}$. This procedure leads to the following relation:

$$
\begin{aligned}
[I(\check{q})]^{\diamond} &= \left[\frac{1}{\check{q}^{2^*} - \check{q}^{-2^*}} \frac{e^{i\frac{\pi}{4}\operatorname{sign}\left(\frac{H}{P}\right)}}{2\sqrt{2K}} \sqrt{\left|\frac{H}{P}\right|} \int\limits_{\substack{-\infty \\ [\beta=0]}}^{+\infty} d\beta\, \check{q}^{-\frac{1}{4}\frac{H}{P}\beta^2} \frac{\prod_{j=1}^{N}\left(\check{q}^{-\frac{\beta}{2p_j}} - \check{q}^{\frac{\beta}{2p_j}}\right)}{\left(\check{q}^{-\frac{\beta}{2}} - \check{q}^{\frac{\beta}{2}}\right)^{N-2}} \right]^{\vee} \qquad (4.49) \\
&= \left[\frac{1}{\sqrt{\frac{2}{K}}\sin\left(\frac{\pi}{K}\right)} \frac{\operatorname{sign}\left(\frac{H}{P}\right)}{K} e^{i\frac{3}{4}\pi\operatorname{sign}\left(\frac{H}{P}\right)} \sqrt{\left|\frac{H}{P}\right|} \int\limits_{\substack{-\infty \\ [\beta=0]}}^{+\infty} d\beta\, e^{-\frac{i\pi}{2K}\frac{H}{P}\beta^2} \frac{\prod_{j=1}^{N}\sin\left(\frac{\pi}{K}\frac{\beta}{p_j}\right)}{\sin^{N-2}\left(\frac{\pi}{K}\beta\right)} \right]^{\vee}.
\end{aligned}
$$

Since

$$
\left[\check{q}^{4^*P^*H - 3\cdot 4^*\operatorname{sign}\left(\frac{H}{P}\right) - 3\sum_{j=1}^{N} s^{\vee}(q_j,p_j)}\right]^{\diamond} = \left[\check{q}^{\frac{1}{4}\frac{H}{P} - \frac{3}{4}\operatorname{sign}\left(\frac{H}{P}\right) - 3\sum_{j=1}^{N} s(q_j,p_j)}\right]^{\vee} \qquad (4.50)
$$

and according to [11], [12], [16] (see, e.g. eq.(4.9) of [16])

$$
\begin{aligned}
Z^{(\mathrm{tr})}(X;k) &= \frac{e^{i\pi\frac{3}{4}\operatorname{sign}\left(\frac{H}{P}\right)}}{K} \frac{\operatorname{sign}(P)}{\sqrt{|P|}} e^{\frac{i\pi}{2K}\left[\frac{H}{P} - 3\operatorname{sign}\left(\frac{H}{P}\right) - 12\sum_{j=1}^{N} s(q_j,p_j)\right]} \\
&\quad \times \int\limits_{\substack{-\infty \\ [\beta=0]}}^{+\infty} d\beta\, e^{-\frac{i\pi}{2K}\frac{H}{P}\beta^2} \frac{\prod_{j=1}^{N}\sin\left(\frac{\pi}{K}\frac{\beta}{p_j}\right)}{\sin^{N-2}\left(\frac{\pi}{K}\beta\right)}, \qquad (4.51)
\end{aligned}
$$

we conclude that eq. (1.23) holds for Seifert manifolds which are rational homology spheres.

5 Discussion

As we have already mentioned in the Introduction, all results of this paper follow rigorously from the "physical" input: Propositions 1.2 and 1.3. The Proposition 1.3 might be harder to prove with mathematical rigor. Even its formulation uses the asymptotic structure (1.16) of Witten's invariant at large values of K. To our knowledge, this structure has not been rigorously derived yet from the surgery formula (1.3). The Proposition 1.2 follows from the properties of Reshetikhin's formula [19] of the Jones polynomial of a link and seems to have better chances for legitimization.

If we forget for a moment about the trivial connection contribution to Witten's invariant, then we can use eq. (1.27) as a definition of the manifold invariants $S_n(M)$. The Proposi-

tion 1.2 gives us enough information in order to derive eq. (1.23) in the form

$$\left[|H_1(M,\mathbb{Z})|\left(\frac{|H_1(M,\mathbb{Z})|}{K}\right)Z'(M;k)\right]^{\diamond} \tag{5.1}$$
$$= \left[|H_1(M,\mathbb{Z})|^{\frac{3}{2}} \exp\left((S_n(M)-S_n(S^3))\left(\frac{i\pi}{K}\right)^n\right)\right]^{\vee}.$$

This also proves Ohtsuki's Theorem 1.2, since we can define the invariants $\lambda_n(M)$ by eq. (1.20). According to the comments of [7], [8], we may also conclude that $S_n(M)$, as defined by the surgery formula (1.27), are indeed invariants of M.

In our previous paper [18] we studied the properties of $S_n(M)$ as they follow from eq. (1.27), Proposition 1.2 and some other properties of Reshetikhin's formula [19]. Now we use eq. (1.20) in order to extend them to $\lambda_n(M)$:

Proposition 5.1 *The invariants $\lambda_n(M)$ are finite type invariants of RHS as defined in [20] and [21]. An invariant $\lambda_n(M)$ is of Ohtsuki order $3n$, Ohtsuki$'$ ([18]) order $2n$ and at most of Garoufalidis order n. Also*

$$2^{4n}\, n!\,(2n)!\,(9n)!\,|H_1(M,\mathbb{Z})|^n\lambda_n(M)\in\mathbb{Z}, \tag{5.2}$$

$$|H_1(M,\mathbb{Z})|^n\lambda_n(M)\in\mathbb{Z}\left[\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{2n}\right]. \tag{5.3}$$

To see that $\lambda_n(M)$ is of exactly Ohtsuki order $3n$ one has to find an n-component link $\mathcal{L}$ such that the alternating sum

$$\sum_{\mathcal{L}'\subset\mathcal{L}}(-1)^{\#\mathcal{L}'}\lambda_n(\chi_{\mathcal{L}'}(S^3)) \tag{5.4}$$

is non-zero (here $\chi_{\mathcal{L}}(S^3)$ denotes a manifold (RHS) constructed by $(1,1)$ surgeries on the components of a link $\mathcal{L}$ in S^3). Recall that according to eq. (1.20)

$$\lambda_n(M)=\sum_{\substack{m_1,\ldots,m_n\geq 0\\ m_1+2m_2+\cdots+nm_n=n}} C^n_{m_1,\ldots,m_n} S_1^{m_1}(M)\cdots S_n^{m_n}(M) \tag{5.5}$$

for some numbers $C_{m_1,\ldots,m_n}$. Consider the n-loop diagram consisting of $(n-1)$ small loops sitting on one big loop, and the corresponding [20] link $\mathcal{L}$. This diagram has no subdiagrams

with only trivalent vertices. Then according to [18], for any $S_{n'}(M)$, $n' < n$ the alternating sums are equal to zero

$$\sum_{\mathcal{L}'' \subset \mathcal{L}'} (-1)^{\#\mathcal{L}''} S_{n'}(\chi_{\mathcal{L}''}(M)) = 0, \tag{5.6}$$

if $\mathcal{L}' \subset L$ and $\#\mathcal{L}' \geq n'$. Therefore an invariant $S_{n'}(M)$ is of Ohtsuki order less than n' with respect to $\mathcal{L}$ and its subdiagrams (of course, there exist other links for which the *l.h.s.* of eq. (5.6) is non-zero if $\#\mathcal{L}' = n'$). As a result, only the term $C_{0,\ldots,0,1} S_n(M)$ in eq. (5.5) matters for the calculation of the alternating sum (5.4). Since, according to [18], the sum

$$\sum_{\mathcal{L}' \subset \mathcal{L}} (-1)^{\#\mathcal{L}'} S_n(\chi_{\mathcal{L}'}(S^3)) \tag{5.7}$$

is non-zero, we conclude that the sum (5.4) is also non-zero. Hence $\lambda_n(M)$ is of exactly Ohtsuki order $3n$.

Finally, let us comment on the relation (1.23) between the invariant $Z'(M;k)$ at prime values of K and the trivial connection contribution to Witten's invariant. This relation is not an obvious result in the sense that the contributions of non-trivial connections to $Z'(M;k)$ do not seem to cancel at prime K (one might expect that there is some cancelation leaving only the contribution of the trivial connection). This can be seen at the example of a lens space $L_{p,q}$ for which

$$Z'(L_{p,q}) = \mathrm{sign}\,(p) \left(\frac{|p|}{K}\right) e^{\frac{6\pi i}{K} s^{\vee}(q,p)} (-1)^{p^*+1} \frac{\sin\left(\frac{\pi}{K} p^*\right)}{\sin\left(\frac{\pi}{K}\right)}, \tag{5.8}$$

$$\frac{Z^{(\mathrm{tr})}(L_{p,q};k)}{Z(S^3;k)} = \frac{\mathrm{sign}\,(p)}{\sqrt{|p|}} e^{\frac{6\pi i}{K} s(q,p)} \frac{\sin\left(\frac{\pi}{K}\frac{1}{p}\right)}{\sin\left(\frac{\pi}{K}\right)}. \tag{5.9}$$

Although these two expressions have many common features, which ultimately lead to the relation (1.23), still their numerical values are quite different. The p and K dependence of $Z'(L_{p,q})$ is somewhat typical of contributions of $U(1)$-reducible connections. Besides, the Dedekind sum $s(q,p)$ is generally a fraction, so $s^{\vee}(q,p) \neq s(q,p)$.

We established the relation (1.23) by comparing directly the surgery formulas. It would be much better to have a conceptual explanation for this phenomenon. One might speculate that it would come from number theory and perhaps p-adic quantum field theories.

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